Contents

GAUGE THEORY, TOM PARKER'S CLASS

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The following is 80% notes from Tom Parker's spring 2019 class on gauge theory, and maybe 20% my own clarification/worked out exercises.

1. Introduction to Gauge Theory

1.1. **Connections, Curvature, and Gauge.**

Fix a complex vector bundle E of rank k over a manifold M of dimension n. We will assume a Riemannian metric g on M whenever required.

$$\begin{array}{c} E \\ {\scriptstyle \varphi}\big\uparrow\big\downarrow \\ (M, g) \end{array}$$

We have a vector space $\Gamma(E)$ of sections φ. To do calculus on E, we will need two types of geometric structures:

(1) A (hermitian) metric h on E,
  i.e. a hermitian inner product on each fiber E_x that varies smoothly with $x \in M$.
  This gives us a map
$$\Gamma(E) \times \Gamma(E) \to C^\infty(M, \mathbb{C})$$
  that is smooth, nonnegative: $\langle \varphi, \varphi \rangle \geq 0$, and is linear/anti-linear over functions in $C^\infty(M; \mathbb{C})$.

(2) A way of defining directional derivatives: $\nabla_X \varphi$, for φ in the direction of X, a vector field on M.

But the usual difference quotient formula

$$\lim_{t \to 0} \frac{\varphi(\gamma(t)) - \varphi(\gamma(0))}{t}$$

doesn't make sense because the vector spaces $E_{\gamma(t)}$ and $E_{\gamma(0)}$ are distinct. We *could* identify the two vector spaces via some linear isomorphism and then subtract, but in general our derivative result will depend on this choice, and there are many non-canonical choices. Instead we define a new object via axioms (note we use the notion of E-valued p-forms, defined by $\Omega^p(M, E) = \Gamma\left(\bigwedge^p T^*M \otimes E\right)$):

Definition 1.1. A (hermitian) connection on $(E, \langle , \rangle)$ is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) =: \Omega^1(M, E)$$

- that satisfies the Leibniz rule: for $f \in C^\infty(M), \varphi \in \Gamma(E)$,

$$\nabla(f\varphi) = \mathrm{d}f \otimes \varphi + f\nabla\varphi$$

- and is compatible with the metric on E: for $\varphi, \psi \in \Gamma(E)$

$$\mathrm{d}\langle\varphi, \psi\rangle = \langle\nabla\varphi, \psi\rangle + \langle\varphi, \nabla\psi\rangle$$

Remark. One can see this last condition as $\nabla h = 0$, as we can define ∇h by the very reasonable

$$\mathrm{d}(h(\varphi, \psi)) = (\nabla h)(\varphi, \psi) + h(\nabla\varphi, \psi) + h(\varphi, \nabla\psi).$$

Remark. A connection gives us a directional derivative. For a vector field X on M, we just plug in X into the 1-form part of $\nabla\varphi$. We write

$$\nabla_X\varphi := (\nabla\varphi)(X) \in \Gamma(E)$$

and ∇ is also called a covariant derivative.

Remark. If we plug X into ∇ right away, we can see it as an operator

$$\nabla_X : \Gamma(E) \to \Gamma(E).$$

Note that after choosing a direction in which to differentiate, ∇_X keeps objects the same type.
It is also useful to see the conversion between the T^*M and the TM parts of a definition like this. Remember that $\mathrm{Vect}(M) = \Gamma(TM)$. We could write

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

and we should specify that it is $C^\infty(M)$-linear in the $\Gamma(TM)$ part (but only $\mathbb{C}$-linear in the $\Gamma(E)$ part).

Lemma 1.2.

(a) *If ∇ and ∇' are connections, then $A = \nabla - \nabla'$ is a 1-form with values in the bundle of skew-hermitian endomorphisms of E, i.e. $A \in \Omega^1(M, \mathrm{End}(E))$ such that, for every $X \in \mathrm{Vect}(M)$*

$$\langle A_X\varphi, \psi\rangle + \langle\varphi, A_X\psi\rangle = 0$$

(b) *Conversely, if ∇ is a connection, and A is as above, then $\nabla' = \nabla + A$ is also a connection.*

Proof.

(a) The Liebniz rule shows that $A(f\varphi) = fA(\varphi)$, so A is a tensor. Then writing $\mathrm{d}\langle\varphi, \psi\rangle$ twice and subtracting shows the skew-hermitian property. Written out:

$$(\nabla - \nabla')(f\varphi) = \mathrm{d}f \otimes \varphi + f\nabla\varphi - \mathrm{d}f \otimes \varphi - f\nabla'\varphi = f(\nabla - \nabla')\varphi.$$

Then

$$0 = \mathrm{d}\langle\varphi, \psi\rangle - \mathrm{d}\langle\varphi, \psi\rangle = \langle(\nabla - \nabla')\varphi, \psi\rangle + \langle\varphi, (\nabla - \nabla')\psi\rangle.$$

(b) The above basically works backwards: A is tensorial, so

$$\begin{aligned}(\nabla + A)(f\varphi) &= \mathrm{d}f \otimes \varphi + f\nabla\varphi + A(f\varphi)\\ &= \mathrm{d}f \otimes \varphi + f\nabla\varphi + fA\varphi\\ &= \mathrm{d}f \otimes \varphi + f(\nabla + A)\varphi.\end{aligned}$$

Metric compatibility follows from

$$\begin{aligned} \mathrm{d}\langle \varphi, \psi\rangle &= \langle \nabla\varphi, \psi\rangle + \langle \varphi, \nabla\psi\rangle \\ &= \langle \nabla\varphi, \psi\rangle + \langle \varphi, \nabla\psi\rangle + \langle A\varphi, \psi\rangle + \langle \varphi, A\psi\rangle \\ &= \langle (\nabla + A)\varphi, \psi\rangle + \langle \varphi, (\nabla + A)\psi\rangle. \end{aligned}$$

□

Example 1.3. For the trivial vector bundle $E = M \times \mathbb{C}^k$, each section φ is a just a map $M \to \mathbb{C}^k$, so $\varphi = (\varphi_1, \dots, \varphi_k)$. Then we can define a standard metric

$$\langle \varphi, \psi\rangle = \sum_i^k \varphi_i \overline{\psi}_i$$

or more generally, any hermitian matrix $h = (h_{ij})$ defines an inner product

$$\langle \varphi, \psi\rangle_h = \sum_{i,j}^k h_{ij}\varphi_i\overline{\psi}_j.$$

Similarly, a standard connection

$$\nabla^0\varphi = (\mathrm{d}\varphi_1, \cdots, \mathrm{d}\varphi_k).$$

But also $\nabla = \nabla^0 + A$, where $A = \sum A_i \mathrm{d}x^i$ with skew-hermitian matrices A_i, is a connection. We think of the matrices A_i as in $\mathrm{End}(\mathbb{C}^k)$.

Lemma 1.4.

(a) *Every vector bundle E admits a hermitian metric*
(b) *Every hermitian vector bundle $(E, \langle\ ,\ \rangle)$ admits a hermitian connection.*

Exercise 1.5. (the proof) Use a partition of unity ρ_α on a locally trivializing open cover of M. Then on each trivial piece, use the standard metric/connection. Note $\sum \rho_\alpha\langle \varphi, \psi\rangle_\alpha$ is still a metric, and $\sum \rho_\alpha \nabla_\alpha$ is still a connection.

Proof. As above, let U_α be a trivializing open cover of M, ρ_α a partition of unity suboordinate to U_α. Define $\langle\ ,\ \rangle = \sum \rho_\alpha \langle\ ,\ \rangle_\alpha$ and $\nabla = \sum \rho_\alpha \nabla^0_\alpha$, using the trivial metric and connection, in the following sense:

(a) If φ, ψ are sections of E, then write $\varphi = \varphi^i e_i,\ \psi = \psi^i e_i$ with the local frame e_i associated to U_α. Then $\langle \varphi, \psi\rangle_\alpha = \sum_i \varphi^i \overline{\psi^i}$. This is still smooth after multiplying by ρ_α and adding. The metric is nonnegative because the standard metric is and $\rho_\alpha \geq 0$. It is hermitian and C^∞-linear because the standard metric is as well. (Note: it doesn't seem like the "unity" matters much here.)

$$\sum_\alpha \rho_\alpha \langle f\varphi, \psi\rangle_\alpha = \sum_\alpha \rho_\alpha f \langle \varphi, \psi\rangle_\alpha = f \sum_\alpha \rho_\alpha \langle \varphi, \psi\rangle_\alpha$$

(b) If $\varphi = \sum_i \varphi^i e_i$, then $\nabla\varphi = \sum_\alpha \rho_\alpha \sum_i (\mathrm{d}\varphi^i)e_i$. This is still C^∞-linear in the vector part:

$$\nabla_{fX} = \sum_\alpha \rho_\alpha \mathrm{d}_{fX} = \sum_\alpha \rho_\alpha f \mathrm{d}_X = f \sum_\alpha \rho_\alpha \mathrm{d}_X$$

and satisfies the Liebniz rule: (Note the partition of unity is important here.)

$$\nabla(f\varphi) = \sum_\alpha \rho_\alpha \sum_i \mathrm{d}(f\varphi^i) \otimes e_i = \sum_\alpha \rho_\alpha \left((\mathrm{d}f) \otimes \varphi + f \sum_i (\mathrm{d}\varphi^i) \otimes e_i \right) = (\mathrm{d}f)\varphi^i + f\nabla\varphi$$

□

The conclusion is that, for a fixed hermitian vector bundle, there is an (infinite dimensional) affine space of connections

$$\mathcal{A} = \{\text{all connections on } (E, \langle\ ,\ \rangle)\} \cong \Omega^1(M, \mathrm{End}^{\mathrm{sk}}(E))$$

given by the affine structure $\nabla \leftrightarrow \nabla^0 + A$ around some chosen origin ∇^0. There is no distinguished "zero" connection, we regard all connections equal.

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• A notation that will make more sense later, we write $\mathfrak{g}_E$ for the bundle $\mathrm{End}^{\mathrm{sk}}(E)$ of skew-adjoint endomorphisms of E. So $\mathfrak{g}_E \subset E \otimes E^* \cong \mathrm{End}(E)$. Also write $\Omega^k(E)$ to mean $\Omega^k(M, E)$ for any bundle $E \to M$.

Definition 1.6. The <u>curvature</u> of a connection ∇ on E is defined by:
For $X, Y \in \mathrm{Vect}(M), \varphi \in \Gamma(E)$,

$$F^{\nabla}_{X,Y}\varphi := \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)\varphi$$

We sometimes just write F.

Lemma 1.7. $F^{\nabla} \in \Omega^2(\mathfrak{g}_E)$

Exercise 1.8. (the proof) Check, by computation, that

(1) F is tensorial in X
(2) F is skew in X, Y (so that $F \in \Omega^2(...)$)
(3) F is tensorial in φ, (so that $F_{X,Y}$ is a bundle endomorphism, i.e. $F \in \Omega^2(\mathrm{End}(E))$)
(4) Differentiate the metric compatibility equation to show that $F_{X,Y}$ is skew adjoint.

Proof. (4) Since F is tensorial in X, Y, it suffices to compute at the point p. For vectors X, Y at a point p, we can extend to vector fields with $[X, Y]_p = 0$.

$$\begin{aligned}
\langle\nabla_X\nabla_Y\varphi, \psi\rangle &= X\langle\nabla_Y\varphi, \psi\rangle - \langle\nabla_Y\varphi, \nabla_X\psi\rangle \\
&= XY\langle\varphi, \psi\rangle - X\langle\varphi, \nabla_Y\psi\rangle - Y\langle\varphi, \nabla_X\psi\rangle + \langle\varphi, \nabla_Y\nabla_X\psi\rangle \\
\implies \langle\nabla_X\nabla_Y\varphi, \psi\rangle - \langle\varphi, \nabla_Y\nabla_X\psi\rangle &= XY\langle\varphi, \psi\rangle - X\langle\varphi, \nabla_Y\psi\rangle - Y\langle\varphi, \nabla_X\psi\rangle \\
\langle\nabla_Y\nabla_X\varphi, \psi\rangle - \langle\varphi, \nabla_X\nabla_Y\psi\rangle &= YX\langle\varphi, \psi\rangle - Y\langle\varphi, \nabla_X\psi\rangle - X\langle\varphi, \nabla_Y\psi\rangle \\
\implies \langle F_{X,Y}\varphi, \psi\rangle + \langle\varphi, F_{X,Y}\psi\rangle &= [X, Y]\langle\varphi, \psi\rangle - X\langle\varphi, \nabla_Y\psi\rangle - Y\langle\varphi, \nabla_X\psi\rangle \\
&\quad + Y\langle\varphi, \nabla_X\psi\rangle + X\langle\varphi, \nabla_Y\psi\rangle = 0
\end{aligned}$$

□

Remark. If (3) holds, then for $p \in M$, let $I_p := \{\varphi \in \Gamma(E) : \varphi(p) = 0\}$. Then

$$\Gamma(E)/I_p \cong E_p \implies \begin{array}{ccc} \Gamma(E) & \xrightarrow{F_{X,Y}} & \Gamma(E) \\ \downarrow & & \downarrow \\ E_p & \longrightarrow & E_p \end{array} \text{ descends to a fiberwise linear map}$$

Definition 1.9. A gauge transformation γ of E is an invertible vector bundle endomorphism, covering the identity (fiber-preserving) and metric preserving.
So the diagram commutes,

$$\begin{array}{ccc} E & \xrightarrow{\gamma} & E \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

for each p, $\gamma_p : E_p \to E_p$ is a complex linear isomorphism, and $\langle \gamma\varphi, \gamma\psi \rangle = \langle \varphi, \psi \rangle$ for all sections.

Example 1.10. Let $L \to M$ be a complex line bundle. Then each real valued function f on M defines a gauge transformation $\gamma = \exp(if)$.

Definition 1.11. The gauge group $\mathcal{G}$ is the group, under composition, of gauge transformations $\gamma : E \to E$.

Locally, on U around $p \in M$, if we choose an orthonormal (ON) frame of E: $\{e_i \in \Gamma(E)\}$ (dual frame: $\{e^i \in \Gamma(E^*)\}$, then any $\gamma \in \mathcal{G}$ has the form

$$\gamma(x) = \sum U_{ij}(x)\ e^i \otimes e_j$$

where U_{ij} is a unitary-matrix-valued function on U. It follows that $\mathcal{G}$ is an infinite dimensional Lie group.

In addition to acting on sections of E, the gauge group also acts on the space of connections by

$$\nabla \mapsto \nabla^\gamma$$
$$\nabla^\gamma \varphi = \gamma \nabla(\gamma^{-1}\varphi) \tag{1}$$

Exercise 1.12. Verify that ∇^γ is a connection.

Proof. It is clear that ∇^γ maps into the right place. We need to check the Leibniz rule. Keep in mind the γ acts only on the E part. At a point $p \in M$, $\gamma_p : E_p \to E_p$ is a linear map, so $\gamma_p(f(p)\varphi_p) = f(p)\gamma_p(\varphi_p)$.

$$\nabla^\gamma(f\varphi) = \gamma\nabla(\gamma^{-1} f\phi) = \gamma\nabla(f\gamma^{-1}\varphi) = \gamma\left(\mathrm{d}f \otimes \gamma^{-1}\varphi + f\nabla(\gamma^{-1}\varphi)\right) = \mathrm{d}f \otimes \varphi + f\nabla^\gamma\varphi$$

□

The intuition of gauge theory is to regard ∇ and all of its transforms ∇^γ as equivalent.

Note that

$$\begin{aligned} \nabla^\gamma_X \nabla^\gamma_Y &= \left(\gamma \circ \nabla_X \circ \gamma^{-1}\right) \circ \left(\gamma \circ \nabla_Y \circ \gamma^{-1}\right) \\ &= \gamma \circ \nabla_X \nabla_Y \circ \gamma^{-1} \end{aligned}$$

So curvature transforms as

$$F^{\gamma}_{X,Y} = \gamma \circ F_{X,Y} \circ \gamma^{-1}. \tag{2}$$

A connection on E induces connections on other bundles:

- On $E \otimes E$ by $\nabla(\varphi \otimes \psi) = \nabla\varphi \otimes \psi + \varphi \otimes \nabla\psi$
- On E^* by $(\nabla\alpha)(\varphi) = d\,(\alpha(\varphi)) - \alpha(\nabla\varphi)$
- On $\mathfrak{g}_E \subset E \otimes E^*$ by combining these.

Definition 1.13. The connection ∇ extends to the following chain

$$\Gamma(E) = \Omega^0(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^1(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^2(E) \xrightarrow{\mathrm{d}^\nabla} \cdots$$

The operator d^∇, sometimes called the exterior covariant derivative, is uniquely determined by the properties:

- $\mathrm{d}^\nabla = \nabla \quad$ on $\Omega^0(E)$.
- $\mathrm{d}^\nabla(\alpha \otimes \omega) = \mathrm{d}\alpha \otimes \omega + (-1)^p \alpha \wedge \mathrm{d}^\nabla \omega \quad$ for $\alpha \in \Omega^p(M)$ and $\omega \in \Gamma(E)$.

Just like the ordinary d, we have a formula for what this looks like in any frame:

$$\mathrm{d}^\nabla = \sum_i e^i \wedge \nabla_{e_i} \tag{3}$$

where $\{e_i\}$ (and $\{e^i\}$) is a local (dual) frame.

Proof. See [Parker's geometry primer] □

Note if $\{f_i\}$ is another local frame, write

$$e_i = \sum_j A^j_i f_j$$

for some matrices $A^j_i(x)$. Then

$$e^i = \sum_j (A^{-1})^i_j f^j$$

and so

$$\sum e^i \wedge \nabla_{e_i} = \sum A^i_j f^j \wedge \nabla_{(A^{-1})^k_i f_k}$$
$$= \sum A^i_j (A^{-1})^k_i f^j \wedge \nabla_{f_k} = \sum f^j \wedge \nabla_{f_j}$$

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Let E be a vector bundle over M. There are three levels of geometric structures on E:

- Metrics
- Covariant derivatives
- Second covariant derivatives. These decompose into
  (i) the covariant Hessian (the symmetric part)

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{D_X Y}$$

Note D is the Levi-Civita connection.

(ii) the curvature (the skew-symmetric part)

$$F_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

On a local chart U, a general vector bundle $E \to M$ looks like $U \times \mathbb{C}^k$. We can choose a basis of sections σ_α so that any section $\varphi \in \Gamma(E)$ can be written, over U, as

$$\varphi = \sum_\alpha \varphi^\alpha \sigma_\alpha.$$

Then, if x^i are local coordinates on U, and $X = \sum X^i \frac{\partial}{\partial x^i}$ is a vector field on U,

$$\begin{aligned}\nabla_X \varphi &= \sum_{\alpha,i} X^i \nabla_{\frac{\partial}{\partial x^i}} (\varphi^\alpha \sigma_\alpha) \\ &= \sum_{\alpha,i} X^i \left(\frac{\partial \varphi^\alpha}{\partial x^i} \sigma_\alpha + \varphi^\alpha \nabla_{\frac{\partial}{\partial x^i}} \sigma_\alpha \right)\end{aligned}$$

Definition 1.14. Define the <u>connection 1-form</u> of ∇ in this trivialization $\{\sigma_\alpha\}$ by

$$\nabla_{\frac{\partial}{\partial x^i}} \sigma_\alpha = \sum_\beta A^\beta_{i\alpha} \sigma_\beta$$

and

$$A^\beta_\alpha := \sum_i A^\beta_{i\alpha} \mathrm{d}x^i.$$

Then

$$\begin{aligned}\nabla_X \varphi &= \sum_{\alpha,i} \left(X^i \frac{\partial \varphi^\alpha}{\partial x^i} \sigma_\alpha + \sum_\beta \varphi^\alpha A^\beta_{i\alpha} X^i \sigma_\beta \right) \\ &= \sum X^i \left[\frac{\partial \phi^\alpha}{\partial x^i} + A^\alpha_{i\beta} \varphi^\beta \right] \sigma_\alpha\end{aligned}$$

which is the same as

$$\nabla \varphi = (\mathrm{d} + A)\varphi.$$

Here d applies the exterior derivative d individually to each component of the vector φ. This depends on our trivialization. It is the pullback of the standard connection on the trivial local bundle $U \times \mathbb{C}^k$.

Example 1.15. As a special case, let $E \to M$ be the tangent bundle $TM \to M$, with a Riemannian metric g. This of course leads to Riemannian geometry. Let's write this out a little bit. For vector fields X, Y on M, $g(X,Y)$ defines a smooth function on M. In local coordinates we can plug in

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = g_{ij}$$

Then we can write

$$g = \sum g_{ij} dx^i \otimes dx^j$$

The Levi-Civita connection D defines a unique preferred connection associated to g, and gives a way of differentiating tangent vector fields. One often sees the Christoffel symbols $D_{\partial_j}\partial_k = \Gamma^i_{jk}\partial_i$:

$$(D_X Y)^i = X^j(\partial_j Y^i + \Gamma^i_{jk} Y^k).$$

This is the connection form.
The Riemann curvature tensor R is defined as in definition 1.6 so the notions of curvature align of course.

1.2. **Parallel Transport.**

Definition 1.16. The choice of a connection ∇ determines a way of parallel transporting fibers of E along paths in M:

Fix a path $p(t)$ in M from $a = p(0)$ to $b = p(1)$ with tangent vector $T = \dot{p}(t)$. The parallel transport is

$$P_t : E_a \to E_{p(t)},$$

the linear isomorphism defined by: Choose $\{\sigma_1, \cdots, \sigma_k\}$ of $E_p(0)$. There exists a unique solution of

$$\nabla_T \sigma_\alpha(t) = 0, \quad \sigma_\alpha(0) = \sigma_\alpha, \quad \alpha = 1, \ldots, k.$$

Then

$$P_t\left(\sum \varphi^\alpha \sigma_\alpha\right) := \sum \varphi^\alpha \sigma_\alpha(t)$$

so $\nabla_T P_t = 0$.

Proof of existence & uniqueness. Locally, we have coordinates $\{x^i\}$ and basis $\{\tau_\alpha\}$ (so $\sigma_\alpha = \varphi(t)^\beta_\alpha(t)\tau_\beta$). Then

$$0 = \nabla_T \sigma_\alpha(t) = \sum T^i \left(\frac{\partial \varphi^\beta_\alpha(t)}{\partial x^i} + A^\beta_{i\mu}\varphi^\mu_\alpha(t)\right)\tau_\beta$$

is a system of first-order ODE's, which we can solve for small time t. □

We can also go backwards: Given P_t, we can recover ∇ by: given X and $a \in M$, choose a path $p(t)$ at a with velocity X at $t = 0$. Then define

$$\nabla_X \varphi = \lim_{t\to 0} \frac{P_{-t}\left(\varphi(p(t))\right) - \varphi\left(p(0)\right)}{t}$$

Thus ∇_X is "infinitesimal parallel transport". (Check: this formula)

Caution: In general, P_t depends on the path $p(t)$.

Lemma 1.17. *For any connection,*
(a) *Parallel transport is an isometry.*
(b) *For all $\varphi \in \Gamma(E)$,*

$$\big|\mathrm{d}|\varphi|\big| \le \big|\nabla\varphi\big|$$

Exercise 1.18. Prove the lemma. Hint: Use metric compatibility on the following
(a) Let $\{\sigma_\alpha\}$ be an ON basis of $E_{p(0)}$ and $\sigma_\alpha(t) = P_t(\sigma_\alpha)$. Then $\frac{\mathrm{d}}{\mathrm{dt}}\langle\sigma_\alpha(t), \sigma_\beta(t)\rangle = \ldots$
(b) $\mathrm{d}|\varphi|^2 = \ldots$, but also $\mathrm{d}\langle\varphi, \varphi\rangle = \ldots$

Or for (b), one can use polar coordinates $\varphi = |\varphi|\xi$, so

$$|\nabla\varphi|^2 = \left|\mathrm{d}|\varphi|\right|^2|\xi|^2 + |\varphi|^2|\nabla\xi|^2 \geq \left|\mathrm{d}|\varphi|\right|^2$$

(So we can see that it is exactly the angular part that drops out.)

Proof.

(a) Say the path has direction X. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_\alpha(t), \sigma_\beta(t)\rangle = \langle\nabla_X\sigma_\alpha(t), \sigma_\beta(t)\rangle + \langle\sigma_\alpha, \nabla_X\sigma_\beta(t)\rangle = 0.$$

($\nabla_X\sigma(t) = 0$ by the definition of parallel transport.) Thus the metric value is constant along the path.

(b) We will show the non-polar proof.

$$2|\varphi|\,(\mathrm{d}|\varphi|) = \mathrm{d}|\varphi|^2 = \mathrm{d}\langle\varphi,\varphi\rangle = 2\langle\varphi,\nabla\varphi\rangle \leq 2|\varphi||\nabla\varphi|$$
$$\mathrm{d}|\varphi| \leq |\nabla\varphi|$$

(In both proofs we seem to skip over $|\varphi| = 0$. But $\mathrm{d}|\varphi|$ may not exist here.)

□

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1.3. **Changing connections.**

- Recall that a connection ∇ on a vector bundle E extends to

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^2(E) \xrightarrow{\mathrm{d}^\nabla} \dots \tag{4}$$

We also have an induced connection $\nabla^{\mathrm{End}(E)}$ on the bundle $\mathrm{End}(E)$, the restriction to $\mathfrak{g}_E$ we call $\nabla^{\mathfrak{g}}$, thinking of $\mathfrak{g}_E \subset \mathrm{End}(E)$. (Eventually we will just use ∇ and d for everything.) So we have

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{\nabla^{\mathfrak{g}}} \Omega^1(\mathfrak{g}_E) \xrightarrow{\mathrm{d}^{\nabla^{\mathfrak{g}}}} \Omega^2(\mathfrak{g}_E) \xrightarrow{\mathrm{d}^{\nabla^{\mathfrak{g}}}} \dots \tag{5}$$

Also recall that curvature $F^\nabla \in \Omega^2(\mathfrak{g}_E)$.

Lemma 1.19.

(a) $(\mathrm{d}^\nabla)^2 = F^\nabla$ *in the sense that, for every* $\omega \in \Omega^p(E)$, $\mathrm{d}^\nabla\mathrm{d}^\nabla\omega = F^\nabla \wedge \omega$ *(so (4) is not a complex).*

(b) $\mathrm{d}^{\nabla^{\mathfrak{g}}}F^\nabla = 0$, *this is called the (2nd or differential) Bianchi identity.*

Note. In (a), $F^\nabla \wedge \omega$ means that we wedge as forms, and also contract the $\mathfrak{g}_E$ and E parts, using (the matrix multiplication) $\mathrm{End}(E) \otimes E \to E$. So in local coordinates, if we write

$$F^\nabla = \sum F^\nabla_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j \in \Omega^2(\mathfrak{g}_E) \quad \text{and} \quad \omega = \sum \omega_I \mathrm{d}x^I \in \Omega^p(E)$$

where each F^∇_{ij} is (a matrix) in $\mathfrak{g}_E$, and each ω_I is (a vector) in E, then $F^\nabla_{ij}\omega_I$ is (a vector) in E, with components $\sum_{\beta=1}^k \left(F^\nabla_{ij}\right)^\alpha_\beta (\omega_I)^\beta$. Thus

$$F^\nabla\omega_I = \sum(F^\nabla_{ij}\omega_I)\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I \in \Omega^{p+2}(E)$$

In (b), note that we are using $\mathrm{d}^{\nabla^{\mathfrak{g}}}$ on $\Omega^\bullet(\mathfrak{g}_E)$ and not d^∇ on $\Omega^\bullet(E)$. In other words we are **not** saying that $\left(\mathrm{d}^\nabla\right)^3 = 0$.

Exercise 1.20. (One proof) Locally around $p \in M$, using what is called a "good frame" of TM. Assuming some Riemannian geometry, this is a frame e_i such that $\left(D_{e_i} e_j\right)_p = 0$, and

$$F^\nabla = \sum F^\nabla_{ij} e^i \wedge e^j \in \Omega^2(\mathfrak{g}_E) \quad \text{and} \quad \omega = \sum \omega_I e^I \in \Omega^p(E)$$

Then compute each part using formula (3) for d. (There may need to be a $\frac{1}{2}$ correction factor.)

Proof. (a) By parallel transporting with the Levi-Civita connection D, we can get a good frame e_i as mentioned above. Then at p, $De^i = 0$ and $F_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i$.

$$\begin{aligned}
\left(\mathrm{d}^\nabla\right)^2 \omega &= \left(\sum e^i \wedge \nabla_{e_i}\right)\left(\sum e^j \wedge \nabla_{e_j}\right)\omega \\
&= \sum e^i \wedge \underbrace{(D_{e_i} e^j)}_{=0} \wedge \nabla_{e_j}\omega + \sum e^i \wedge e^j \wedge \nabla_{e_i}\nabla_{e_j}\omega \\
&= \sum_{i<j} e^i \wedge e^j \wedge (\nabla_i \nabla_j - \nabla_j \nabla_i)\,\omega \\
&= \sum_{i<j} F_{ij} \wedge \omega = F \wedge \omega
\end{aligned}$$

(b) The induced operator $\mathrm{d}^{\nabla^{\mathfrak{g}}}$ on $\mathrm{End}(E)$ is given by the commutator

$$\mathrm{d}^{\nabla^{\mathfrak{g}}} F = [\mathrm{d}^\nabla, F] = \mathrm{d}^\nabla (\mathrm{d}^\nabla)^2 - (\mathrm{d}^\nabla)^2 \mathrm{d}^\nabla = 0.$$

Why this is true:
Just for extra exposition, let's look carefully how the induced $\nabla^{\mathfrak{g}}$ and $\mathrm{d}^{\nabla^{\mathfrak{g}}}$ are defined in (5). Let $\varphi \in \Omega^k(\mathfrak{g}_E)$, $\sigma \in \Omega^l(E)$, and let X be a vector field. Then we define $\nabla^{\mathfrak{g}}_X \varphi \in \Omega^k(\mathfrak{g}_E)$ to act on σ by

$$\begin{aligned}
\left(\nabla^{\mathfrak{g}}_X \varphi\right)(\sigma) &= \nabla_X(\varphi(\sigma)) - \varphi(\nabla_X \sigma) \quad \text{for the case } k = l = 0 \\
\left(\nabla^{\mathfrak{g}}_X \varphi\right) \wedge \sigma &= \nabla_X(\varphi \wedge \sigma) - \varphi \wedge (\nabla_X \sigma) \quad \text{in general.}
\end{aligned}$$

(See the above note about the wedge.)
Now $\mathrm{d}^{\nabla^{\mathfrak{g}}}$ is defined using the formula $e^i \wedge \nabla_{e_i}$.

$$\begin{aligned}
\left(\mathrm{d}^{\nabla^{\mathfrak{g}}} \varphi\right) \wedge \sigma &= \left(e^i \wedge \nabla^{\mathfrak{g}}_{e_i} \varphi\right) \wedge \sigma \\
&= e^i \wedge \nabla^{\mathfrak{g}}_{e_i}(\varphi) \wedge \sigma \\
&= e^i \wedge \nabla_{e_i}(\varphi \wedge \sigma) - e^i \wedge \varphi \wedge (\nabla_{e_i}\sigma) \\
&= e^i \wedge \nabla_{e_i}(\varphi \wedge \sigma) - (-1)^k \varphi \wedge (e^i \wedge \nabla_{e_i}\sigma) \\
&= \mathrm{d}^\nabla(\varphi \wedge \sigma) - (-1)^k \varphi \wedge (\mathrm{d}^\nabla \sigma)
\end{aligned}$$

In particular, for the curvature F, $k = 2$, so $\mathrm{d}^{\nabla^{\mathfrak{g}}}$ is given by the commutator and

$$\begin{aligned}
(\mathrm{d}^{\nabla^{\mathfrak{g}}} F) \wedge \sigma &= \mathrm{d}^\nabla(F^\nabla \wedge \sigma) - F^\nabla \wedge (\mathrm{d}^\nabla \sigma) \\
&= \mathrm{d}^\nabla(\mathrm{d}^\nabla \mathrm{d}^\nabla \sigma) - \mathrm{d}^\nabla \mathrm{d}^\nabla(\mathrm{d}^\nabla \sigma) = 0
\end{aligned}$$

□

Lemma 1.21.

(a) *If γ is a gauge transformation on E, then γ acts on ∇ and F by*

$$\nabla^\gamma = \gamma \nabla \gamma^{-1}$$
$$F^\gamma = \gamma F \gamma^{-1} = (\operatorname{Ad}\gamma) F$$

(b) *For $\nabla' = \nabla + A$, where $A \in \Omega^1(\mathfrak{g}_E)$,*

$$F' = F + \mathrm{d}^\nabla A + [A \wedge A] \tag{6}$$

where $[A \wedge A]$ is the $\mathfrak{g}_E$-valued 2-form defined by

$$[A \wedge B]_{X,Y} = \frac{1}{2}([A_X, B_Y] - [A_Y, B_X]) = \frac{1}{2}(A_X B_Y - B_Y A_X - A_Y B_X + B_X A_Y).$$

In particular

$$[A \wedge A]_{X,Y} = [A_X, A_Y] = A_X A_Y - A_Y A_X.$$

(Recall that A_X, B_Y, etc. are like matrices.)

Exercise 1.22. Prove this, by computing with a good frame again. Part (a) was done before.

Proof. For part (a) see (1) and (2).

$$\begin{aligned}
\nabla'_i \nabla'_j \varphi &= (\nabla_i + A_i)(\nabla_j + A_j)\varphi \\
&= \nabla_i \nabla_j \varphi + A_i \nabla_j \varphi + \nabla_i (A_j \varphi) + A_i A_j \varphi \\
&= \nabla_i \nabla_j \varphi + A_i \nabla_j \varphi + (\widetilde{\nabla}_i A_j)\varphi + A_j \nabla_i \varphi + A_i A_j \varphi \\
\nabla'_j \nabla'_i \varphi &= \nabla_j \nabla_i \varphi + A_j \nabla_i \varphi + (\widetilde{\nabla}_j A_i)\varphi + A_i \nabla_j \varphi + A_j A_i \varphi \\
F'_{ij}\varphi &= F_{ij}\varphi + (\widetilde{\nabla}_i A_j - \widetilde{\nabla}_j A_i)\varphi + (A_i A_j - A_j A_i)\varphi
\end{aligned}$$

The operator d^∇ is the antisymmetrization of ∇. Note $\widetilde{\nabla} A \in \Gamma(T^*M \otimes \mathfrak{g})$. □

1.4. **Chern classes.**

Each hermitian vector bundle $E \to M$ of rank k determines de Rham cohomology classes

$$c_l(E) \in H^{2l}(M) \quad \text{for } l = 1, \cdots, k$$

as follows:

Definition 1.23. Choose a connection ∇ on E. The l^{th} chern class is

$$c_l(E) = \left(\frac{i}{2\pi}\right)^l \operatorname{tr}\left(F^\nabla \wedge \cdots \wedge F^\nabla\right) \in \Omega^{2l}(M)$$

where $F^\nabla \wedge \cdots \wedge F^\nabla$ is a $\mathfrak{g}_E$-valued $2l$-form.

Definition 1.24. The total chern class is

$$\begin{aligned}
c(E) &= \det\left(\operatorname{Id} + \frac{i}{2\pi} F^\nabla\right) \\
&= I + c_1(E) + c_2(E) + \cdots + c_k(E).
\end{aligned}$$

Remark. $\det(I + B) = \sum \operatorname{tr} B^k$

Lemma 1.25.
(a) *The form* $c_l(E)$ *is closed, and*
(b) *its cohomology class is independent of* ∇.

Exercise 1.26.
(a) (Check:) Figure out the *correct* statement of the fact:
$$\mathrm{d}[\det M(x)] = \det M(x) \cdot \mathrm{d}(\mathrm{tr} M(x))$$
for a matrix valued function $M(x)$. Use this with the Bianchi identity.
(b) For $\nabla' = \nabla + A$, consider $\nabla^t = \nabla + tA$, with $0 \leq t \leq 1$. Then compute $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} F = \mathrm{d}A$, and the following:
$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} c_l(E, \nabla^t) = l \cdot \mathrm{tr}(\mathrm{d}(A \wedge F \wedge \cdots \wedge F)) = \mathrm{d}\omega_t$$
(actually figure out the problem with the t dependence) where $\omega_t = l \cdot \mathrm{tr}(A \wedge F \wedge \cdots \wedge F)$, showing that
$$c_l(E, \nabla') = c_l(E, \nabla) + \mathrm{d}\eta$$
for $\eta = \int_0^1 w_t \, \mathrm{d}t$.

Example 1.27. A complex line bundle $L \to M$ has a single chern class $c_1(M) \in H^2(M)$.

Example 1.28. A rank 2 complex vector bundle $E \to M$ has $c_1(E) \in H^2(M)$ and $c_2(E) \in H^4(M)$.

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2019/01/22

1.5. **Some relation to physics.**

Example 1.29. Let L be a complex line bundle over $M = \mathbb{R}^{1,3}$ with a hermitian metric. Consider the action functional
$$\Phi(\nabla) = \frac{1}{2} \int_M |F|^2 \, \mathrm{dvol}_g$$
Then looking for critical points gives us
$$\begin{cases} \mathrm{d} \star F = 0 & \text{Vacuum Maxwell Equations} \\ \mathrm{d}F = 0 & \text{2nd Bianchi identity} \end{cases}$$
Recall the Hodge star operator $\star$ on $\bigwedge^p T^*M$.
$$\int_M \alpha \wedge \star\alpha = \int_M |\alpha|^2 \, \mathrm{dvol}_g \tag{7}$$

Example 1.30. Now include a current in the situation above. In physics, this is
$$(\rho, \mathbf{J}) = (\rho, J_1, J_2, J_3) \leftrightarrow \rho \frac{\partial}{\partial t} + \sum J_i \frac{\partial}{\partial x_i}.$$
So we can say current is a vector field on M. But the law of charge conservation implies the continuity equation:
$$\frac{\partial \rho}{\partial t} + \mathrm{div} J = 0 \tag{8}$$

Instead, consider the current 3-form

$$j = -\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z + J_1 \mathrm{d}t\mathrm{d}y\mathrm{d}z - J_2 \mathrm{d}t\mathrm{d}x\mathrm{d}z + J_3 \mathrm{d}t\mathrm{d}x\mathrm{d}y$$

Then (8) becomes

$$\mathrm{d}j = 0 \tag{9}$$

Now let's put this current into (the) action:

$$\Phi(\nabla, j) = \frac{1}{2}\int_M |F|^2 + \int_M j \wedge A$$

Then, (looking for critical points), for a fixed j and varying connection ($\nabla \mapsto \nabla + tB$, $A \mapsto A + tB$, for $B \in \Omega^1(M)$), we find the Euler Lagrange equations:

$$\begin{aligned} 0 = \delta_B \Phi(\nabla, j) &= \int_M \langle F, \mathrm{d}B\rangle + j \wedge B \\ &= \int_M \langle \mathrm{d}^* F, B\rangle - \langle \star j, B\rangle \\ \implies \mathrm{d}^* F - \star j = 0 &\implies \begin{cases} \mathrm{d} \star F = j \\ \mathrm{d}F = 0 \qquad \text{Bianchi} \end{cases} \end{aligned}$$

which are Maxwell's equations with source. (d^* is defined to be the L^2 adjoint of d.)

Remark.
(1) It is not clear yet that $\int j \wedge A$ is well defined (gauge invariant?). But we know that j is a closed 3-form and $H^3(\mathbb{R}^{1,3}) = 0$ so $j = \mathrm{d}k$ for some k. Then

$$\int j \wedge A = \int \mathrm{d}k \wedge A = -\int k \wedge \mathrm{d}A = -\int k \wedge F$$

is well defined.
Also note: If $j = \mathrm{d}k'$,

$$0 = \int \mathrm{d}(k - k') \wedge A = -\int (k - k') \wedge F$$

so $\int k \wedge F$ is independent of k.
(2) The current j can be taken as a given, or regarded as a variable.

Special Case

Consider a moving particle with world line γ. There is an associated current, the distribution 1-form j_γ.

By distribution we mean $j_\gamma \in [\Omega^1(M)]^*$. As a quick example, recall that the δ "function" is a $C^\infty(M)$ distribution defined by $\delta_x(f) = f(x)$.

The associated current is defined by

$$j_\gamma(\alpha) = \int_\gamma \alpha \qquad \forall \alpha \in \Omega^1(M).$$

Then

$$\Phi(A, j_\gamma) = \frac{1}{2}\int_M |F|^2 + \int_\gamma A.$$

This last integral is related to holonomy (= parallel transport).

Recall: For $E \to M$ with local trivialization $\{\sigma_1, \cdots, \sigma_k\} \in \Gamma(E|_U)$. Then $\varphi \in \Gamma(E|_U)$ is $\varphi = \sum \varphi^\alpha(x)\sigma_\alpha$ and $\nabla_{\partial_{x^i}}\sigma_\alpha = \sum A^\beta_{i\alpha}\sigma_\beta$, with $A^\beta_{i\alpha}\mathrm{d}x^i$, a (skew-hermitian) matrix-valued 1-form.

Then the holonomy along a path γ is $\mathcal{H}_\gamma : E_p \to E_q$ defined by $\mathcal{H}_\gamma(t)\varphi_0 = \varphi(t)$. For $\varphi(t)$ a solution,

$$0 = \nabla_{\dot\gamma}\varphi = \sum \left(\frac{\mathrm{d}\varphi^\alpha}{\mathrm{d}t} + A^\alpha_{i\beta}\dot\gamma^i\varphi^\beta \right)\sigma_\alpha$$

$$\frac{\mathrm{d}\varphi^\alpha}{\mathrm{d}t} = -A^\alpha_{i\beta}\dot\gamma^i\varphi^\beta \quad \forall\alpha \tag{10}$$

For E a complex line bundle, we can write A as iA for a real valued 1-form A. Then (10) becomes

$$\dot\varphi = -iA(\dot\gamma)\varphi$$
$$\implies \phi(t) = e^{-i\int_\gamma A}\varphi_0 = \mathcal{H}^\nabla_\gamma(t)\phi_0$$

Hence

$$e^{i\Phi(\nabla, j_\gamma)} = e^{\frac{i}{2}\int_M |F|^2 - i\int_\gamma A} = \mathcal{H}_\gamma e^{\frac{i}{2}\int_M |F|^2}$$

Example 1.31. Examples 1.29 and 1.30 immediately generalize to $E \to M^n$, a rank k complex vector bundle.

$$\Phi(\nabla) = \frac{1}{2}\int_M |F|^2 + \langle j, A\rangle \quad \text{where } j \in \Omega^1_M(\mathfrak{g}_E)$$

The E-L equations are the same

$$\begin{cases} \star\mathrm{d}\star F = j & \text{E-L eq} \\ \mathrm{d}F = 0 & \text{Bianchi} \end{cases}$$

These are Yang-Mills equation with a current.

Note (to self). Maybe this is the one Nöether theorem? First, fix a connection ∇^0. Then, if given a connection $\nabla = \nabla^0 + A$ and a gauge transformation γ, we have another connection, written in two ways

$$\nabla' = \gamma\nabla\gamma^{-1} = \nabla^0 + A'.$$

What is A'? Acting on a section $\varphi \in \Gamma(E)$,

$$\begin{aligned} \nabla'\varphi &= \gamma\left(\nabla^0 + A\right)\gamma^{-1}\varphi \\ &= \gamma\nabla^0(\gamma^{-1}\varphi) + \gamma A\gamma^{-1}\varphi \\ &= \nabla^0\varphi + \gamma(\nabla^0\gamma^{-1})\varphi + \gamma A\gamma^{-1}\varphi \\ &= \nabla^0\varphi - (\nabla^0\gamma)\gamma^{-1}\varphi + \gamma A\gamma^{-1}\varphi \\ \implies A' &= \gamma A\gamma^{-1} - (\nabla^0\gamma)\gamma^{-1} \end{aligned}$$

[—need to finish this, connect the above with below—]

Running the above backwards, assuming gauge invariance of the action gives $\int j \wedge A = \int j \wedge (A + \mathrm{d}B)$ for any B [is this what gauge inv gives? see above calculation]. This implies

$$0 = \int_M j \wedge \mathrm{d}B = -\int_M \mathrm{d}j \wedge B$$

for any B, i.e. that $\mathrm{d}j = 0$[is this true, actually?], the local conservation of charge.

Q: How can we interpret a continuity equation for j?
Again, current is related to holonomy, which satisfies

$$\dot{H}(t) = -A(t)H(t),$$

the solution of which is

$$\mathcal{H}(t) = \mathcal{P}e^{\int A} \quad \text{(the "path-ordered exponential")}$$

i.e.

$$\begin{aligned} A_i(t) &= \text{path in the Lie algebra of } U(k) \\ &= \text{path in } \{\text{left-invariant vector fields}\} \end{aligned}$$

and

$$\mathcal{H}(t) = \text{flow of time-dependent left-invariant vector field on } U(k)$$

found by integrating

$$\begin{aligned} \mathcal{H}(t) &= I - \int_0^t \mathcal{H}(s)^{-1} A(s) \mathrm{d}s \\ &= I - \int_0^t A(s)\mathrm{d}s + \int_0^t \int_0^{t_1} A(s)\mathcal{H}(s')\mathrm{d}s\mathrm{d}s' + \cdots \end{aligned}$$

[For a better recount of parallel transport and path-ordering, see the document handed out in class (very good explanation, for Riemannian case), or Baez & Muniain - Gauge Fields, Knots and Gravity]

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2019/01/25

Definition 1.32. An <u>observable</u> is a function $f : \mathcal{A} \to \mathbb{R}$ or $\mathbb{C}$ that is gauge-invariant:

$$f(\nabla) = f(g\nabla g^{-1}) \quad \forall g \in \mathcal{G}$$

so that it descends to a function on $\mathcal{B} = \mathcal{A}/\mathcal{G}$.

Recall the effects of $g \in \mathcal{G}$
(i) $\nabla \mapsto g\ \nabla\ g^{-1}$
(ii) $F \mapsto g\ F\ g^{-1}$

Definition 1.33. For a path $\gamma(t)$, Parallel transport $P_\gamma : E_p \to E_q$

$$P_\gamma \mapsto g(q)P_\gamma g(p)^{-1}.$$

Note $\nabla_{\dot{\gamma}} s_i = 0 \iff (g\nabla_{\dot{\gamma}} g^{-1})(gs_i) = 0$. In particular, if γ is actually a loop in M, then parallel transport around γ is a unitary endomorphism called <u>Holonomy</u> around γ

$$\mathcal{H}_\gamma(p) \in U(k) \subset \mathrm{End}(E_p)$$

Example 1.34.

$$f(\nabla) = \int_M |F|^2 \mathrm{dvol}_g$$

satisfies $|gFg^{-1}|^2 = |F|^2$.

Example 1.35. Let $M_{\mathbb{C}}(k) = \{k \times k \text{ complex matrices}\}$. An invariant polynomial is a function on $M_{\mathbb{C}}$ such that

- $\tau(A)$ is a polynomial in A_{ij}
- $\tau(gAg^{-1}) = \tau(A)$ for all $g \in U(k)$

One can prove that the ring of invariant polynomials is generated by

$$1, \tau_1(A) = \mathrm{tr}(A), \cdots, \tau_k(A) = \mathrm{tr}(A^k).$$

Hence for each loop γ in M and $p \in \gamma$,

$$f_{k,\gamma}(\nabla) = \tau_k(\mathcal{H}_\gamma(p)) = \mathrm{tr}((\mathcal{H}_\gamma)^k)$$

is $\mathcal{G}$-invariant $\implies$ f is an observable.

Remark (on QFT). For a gauge theory

$$\begin{array}{c} (E, \langle\ ,\ \rangle) \\ \downarrow \\ (M, g) \end{array} \tag{11}$$

with action $\Phi(\nabla) = \frac{1}{2}\int_M |F|^2$. Each observable $f : \mathcal{B} \to \mathbb{R}$ has expectation

$$\langle f \rangle = \frac{\int_{\mathcal{B}} f e^{i\int_M |F|^2} \mathrm{d}V}{\int_{\mathcal{B}} e^{i\int_M |F|^2} \mathrm{d}V}$$

Even though $\mathcal{B}$ is infinite dimensional and there is no such $\mathrm{d}V$, it still works for some reason if you are a physicist.

Remark. It is a theorem that if ∇, ∇' are connections with $f_{k,\gamma}(\nabla) = f_{k,\gamma}(\nabla')$ for every $f_{k,\gamma}$, then ∇ and ∇' are gauge equivalent. Thus $\{f_{k,\gamma} : \gamma \text{ is any loop}\}$ distinguish the points of $\mathcal{B}$.

1.6. **Flat connections.**

Example 1.36. Returning to the gauge theory setup (11) where E is a complex, rank k and M is compact, connected and Riemannian, and where $\Phi(\nabla) = \frac{1}{2}\int_M |F|^2 \mathrm{dvol}_g$. Note that $\Phi(\nabla) \geq 0$ with equality if and only if $F \equiv 0$.

Definition 1.37. A connection ∇ is flat if $F \equiv 0$. ($\implies$ $g.\nabla$ is flat for all $g \in \mathcal{G}$ since $F^{g.\nabla} = gFg^{-1} = 0$.) Then

$$\mathcal{M}_{\text{flat}} = \{\text{flat connections } \nabla\} \big/ \mathcal{G}$$

is one component of the moduli space of Yang-Mills connections.

Remark. $\mathcal{M}_{\text{flat}}$ may not be connected.

Remark. Often there are no flat connections. Note that if $F = 0$, then $\det(I - \frac{i}{2\pi}F) = 1$ but

$$\det(I - \frac{i}{2\pi}F) = 1 + c_1(E) + c_2(E) + \cdots$$
$$\implies \text{all chern classes } c_i(E) = 0$$

Thus if E is a bundle with any nonzero chern class, then E has no flat connections.

Definition 1.38. A pair (E, ∇) is called locally trivial if there exists a parallel local frame, i.e. each point $p \in M$ has a local frame $s_1, \ldots, s_k \in \Gamma(E|_U)$ and $\nabla s_i \equiv 0$ for each i.

This means locally we can do calculations on a trivialization using ∂, the trivial connection, in place of ∇.

Example 1.39 (Möbius band)**.**

$$\begin{array}{ccc} E = S^1 \times \mathbb{R} & & E = [0, 2\pi] \times \mathbb{R} \\ \downarrow & = & \downarrow \\ S^1 & & S^1 \end{array}$$

with fibers E_0 and $E_{2\pi}$ identified by $(0, x) \sim (2\pi, -x)$.
The connection $\nabla = \frac{\partial}{\partial\theta}$ has $s(\theta) = (\theta, \text{const.})$ as a parallel section, and $F \equiv 0$. $\implies$ E is locally flat, but not globally trivial, with holonomy $-\text{Id}$ around S^1.

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2019/01/28

Example 1.40. On $E = S^1 \times \mathbb{C}$ over $M = S^1$, the trivial complex line bundle, define the connection $\nabla = \mathrm{d} + A$ where $A = if(\theta)\mathrm{d}\theta$, a 1×1 skew-hermitian matrix. Let's take $f(\theta) \equiv c \in \mathbb{R}$ (constant). Then $F = 0$ and a section s is parallel iff

$$\nabla s = \left[\frac{\partial s}{\partial\theta} + ics(\theta)\right]\mathrm{d}\theta = 0.$$
$$\text{So, } s(\theta) = Ce^{-ic\theta}$$

Hence the holonomy around S^1 is

$$\mathcal{H}^{\nabla^c} = e^{2\pi ic}$$

So $\mathcal{H} = \text{Id}$ iff $c \in \mathbf{Z}$. Thus there exist nonzero sections s with $\nabla s = 0$ iff $c \in \mathbf{Z}$.

Lemma 1.41.
(a) (E, ∇) *is locally trivial* $\iff$ ∇ *is flat*
(b) *If E admits a flat connection, then*

$$\mathcal{M}_{\text{flat}} := \frac{\{\text{Flat connections}\}}{\mathcal{G}} = \frac{\text{Hom}(\pi_1 M, U(k))}{\text{conj.}}$$

Proof.

(a) If E is locally trivial, there is a local frame $s_1, \ldots, s_k$ such that $\nabla_i s_l = 0$ for all i, l. Then

$$0 = \nabla_i \nabla_j s_l - \nabla_j \nabla_i s_l = F_{ij} s_l \implies F \equiv 0$$

Now suppose $F \equiv 0$.
Fix $p \in M$, local coordinates x^i around p, and and ON basis $s_1(0), \ldots, s_k(0)$ of E_p. Extend each section to a neighborhood of p by parallel transport in the radial direction:
In coordinates (r, θ) on $[0, \varepsilon] \times S^{n-1}$, we have $\nabla_r s_i = 0$ locally. In particular $(\nabla_i s_j)_p = 0$ for all i, j. Hence at (R, θ),

$$\begin{aligned}(\nabla_j s_i)_{(R,\theta)} &= (\nabla_j s_i)_p + \int_0^R \nabla_r (\nabla_j s_i) \mathrm{d}r \\ &= 0 + \int \underbrace{F_{r,j}}_{=0} s_i + \nabla_j \underbrace{(\nabla_r s)}_{=0}\end{aligned}$$

So $\nabla_j s_i = 0$ (locally).
Note: We used that $\nabla_r((\nabla_j s_i)^l s_l) = (\partial_r (\nabla_j s_i)^l) s_l + (\nabla_j s_i)^l \nabla_r s_l$ for the use of FTC, but again $\nabla_r s_l = 0$.

(b) If $F \equiv 0$, then (a) implies the holonomy around all small loops is the identity. Fix $p \in M$ and consider loops λ starting at p (i.e. $\lambda : S^1 \to M$ and $\lambda(1) = p$).
Any homotopy can be broken down into a sequence of homotopies that are each supported in small neighborhoods. Thus $\mathcal{H}_\lambda$ depends only on the homotopy class $[\lambda] \in \pi_1(M, p)$, and in particular $\mathcal{H}_\lambda = \mathrm{Id}$ for contractible loops λ.
Changing the basepoint (from p to q) results in conjugation: Pick a path μ from q to p. Then

$$\mathcal{H}_{\mu^{-1}\lambda\mu} = \mathcal{H}_{\mu^{-1}} \mathcal{H}_\lambda \mathcal{H}_\mu = (\mathcal{H}_\mu)^{-1} \mathcal{H}_\lambda \mathcal{H}_\mu$$

Thus $\mathcal{H}_\lambda$ mod conjugacy depends only on $[\lambda] \in \pi_1 M$.
Conversely, if $\widetilde{M} \to M$ is the universal cover, take $\nabla = \mathrm{d}$ on $E \to \widetilde{M}$ and mod out by $\pi_1 M$. (What does this mean?)

□

Example 1.42. If M is simply-connected, then $\mathrm{Hom}(\pi_1 M, U(k)) = 1$, and so all flat connections are gauge equivalent to the trivial connection on the trivial bundle.

$$\implies \mathcal{M}_{\text{flat}} = \begin{cases} \{\text{pt}\} & \text{if } E \text{ is a trivial bundle} \\ \varnothing & \text{otherwise} \end{cases}$$

Background from linear algebra

Lemma 1.43. *Any $A \in U(k)$ can be diagonalized by a unitary change of basis, i.e. there is a $B \in U(k)$ such that $BAB^{-1} = D$, where*

$$\begin{pmatrix} e^{i\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\lambda_k} \end{pmatrix} \quad \lambda_i \in \mathbb{R}$$

and B is unique up to reordering basis.

Let $T^k = \left\{\operatorname{diag}\left(e^{i\lambda_1}, \cdots, e^{i\lambda_k}\right) : \lambda_i \in \mathbb{R}\right\} = S^1 \times \cdots \times S^1 \subset U(k)$ be the maximal torus. The permutation group S_k acts on T^k. Then Lemma 1.43 tells us, as $U(k)$ acts on itself by conjugation, each orbit intersects T^k a finite number of times in exactly an orbit of S_k on T^k.

Example 1.44. For $M = S^1$ and by Lemma 1.41,

$$\mathcal{M}_{\text{flat}} = \frac{\{\text{flat connections on } E \xrightarrow{\text{rank } k} S^1 \nabla\}}{\mathcal{G}} = \frac{\operatorname{Hom}(\pi_1 M, U(k))}{\text{conj.}} = \frac{\operatorname{Hom}(\mathbf{Z}, U(k))}{\text{conj.}} = \frac{U(k)}{\text{conj.}} = \frac{T^k}{S_k}$$

(Any homomorphism $\mathbf{Z} \to U(k)$ is determined by the image of 1.)
In particular,

(i) For $U(1) = S^1$, $\mathcal{M}_{\text{flat}} \cong S^1 = \{e^{ic\theta}\}$. (Any ∇^c is gauge equivalent.)
(ii) For $U(2)$, $\mathcal{M}_{\text{flat}} \cong T^2/\mathbf{Z}_2$. (Here the action is not free. The matrix $\operatorname{diag}(e^{i\lambda}, e^{i\lambda})$ acts as the identity. $\mathcal{M}_{\text{flat}}$ is an orbifold.)

Example 1.45. Take $M = \mathbb{R}P^3 = S^3/\mathbf{Z}_2$. We have $\pi_1 \mathbb{R}P^3 = \mathbf{Z}^2$. Then

$$\mathcal{M}_{\text{flat}} = \frac{\operatorname{Hom}(\mathbf{Z}_2, U(2))}{\text{conj.}} = \frac{\{\operatorname{diag}(\pm 1, \pm 1)\}}{\text{conj.}} = \frac{\{\operatorname{diag}(\pm 1, \pm 1)\}}{\mathbf{Z}_2} = 3 \text{ points.}$$

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2019/02/01

1.7. **Reducible connections.**

This is another special type of connection.

Definition 1.46. A connection ∇ on $E \to M$ is <u>reducible</u> if there exists a nonzero section $s \in \Gamma(E)$ such that $\nabla s = 0$.

Note. $\mathrm{d}|s|^2 = \mathrm{d}\langle s, s\rangle = 2\langle s, \nabla s\rangle$. So $\nabla s = 0 \implies |s|^2 \equiv \text{constant} \neq 0$.

Lemma 1.47. *For a rank k complex vector bundle $E \to M$, the following are equivalent*

(a) *E admits a reducible connection ∇.*
(b) *$E \cong \mathbb{C} \oplus E'$ for some rank $k-1$ vector bundle E' ("E splits off a line").*
(c) *The top chern class $c_k(E) = 0$ in $H^{2k}(M)$.*

Proof. (a) $\implies$ (b):
Given ∇, and a section $s \neq 0$ with $\nabla s = 0$, define a bundle map φ

$$\begin{array}{ccc} \mathbb{C} \times M & \xrightarrow{\ \varphi\ } & E \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array} \qquad \text{by } \varphi(\lambda, x) = \lambda s(x)$$

This is injective and locally trivial $\implies$ the image of φ is a trivial complex line bundle (a subbundle of E). Now set

$$E' = \{e \in E : \langle e, s\rangle \equiv 0\}$$

After local triviality, this is a rank $k-1$ complex vector bundle. Furthermore, any $\sigma \in \Gamma(E')$ is orthogonal to s, so

$$0 = \mathrm{d}\langle \sigma, s\rangle = \langle \nabla\sigma, s\rangle + \underbrace{\langle \sigma, \nabla s\rangle}_{=0}$$
$$\implies \nabla_X \sigma \in \Gamma(E').$$

Hence, under $E = \mathbb{C} \oplus E'$ the connection splits:

$$\nabla = \begin{pmatrix} d & 0 \\ 0 & \nabla' \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 0 & F' \end{pmatrix} \tag{12}$$

(b) $\implies$ (a):
Any connection ∇' on E' extends to a connection ∇ on E using (12), with a section $s = (1,0)$ that satisfies $\nabla s = 0$, making ∇ reducible.

(b) $\implies$ (c):
The top chern class $c_k(E)$ is represented by

$$c(E, \nabla) = \det\left(I + \frac{i}{2\pi}F\right) = 1 + c_1 + \cdots + c_k$$

and its $2k$-form part

$$c_k(E,\nabla) = \det\left(\frac{i}{2\pi}F\right) = 0 \quad \text{by (12)}$$

(c) $\implies$ (b):
By bundle theory (see Milnor & Stasheff), the top chern class = the euler class, and the euler class $e(E) = 0 \iff E$ admits a nowhere zero section. ($e(E) \in H^{\mathrm{rk}E}(M)$.) □

1.8. More examples of actions.

So far our actions have only involved $|F|^2$. Let's bring in a section $\varphi \in \Gamma(E)$. (A Higgs field?) First, some false starts:

- Let $\Phi(\varphi) = \int_M |\varphi|^2 \mathrm{dvol}_g$. This yields: $\varphi \equiv 0$ as the only field equation.
- In an attempt to get the critical point away from $\varphi = 0$, let $\Phi(\varphi) = \displaystyle\int_M \left(|\varphi|^2 - 1\right)^2 \mathrm{dvol}_g$. The field equations are either $|\varphi|^2 \equiv 1$ or $\varphi \equiv 0$. In the nonzero case, then E splits off as $\mathbb{C} \oplus E'$, where the line is $\mathbb{C} = \mathbb{C}\varphi$, and φ is gauge equivalent to $(1,0) \in \Gamma(E)$. This is not interesting by itself (more on this later).

Example 1.48. Let $D : \Gamma(E) \to \Gamma(E')$ be a 1^{st} order differential operator, and consider

$$\Phi(\varphi) = \frac{1}{2}\int_M |D\varphi|^2 \geq 0$$

The field equations: for $\psi \in C_c^\infty(E)$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\varphi + t\psi)\Big|_{t=0} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M \langle D(\varphi + t\psi), D(\varphi + t\psi)\rangle\Big|_{t=0}$$
$$= \int_M \mathrm{Re}\ \langle D\varphi, D\psi\rangle = \int_M \mathrm{Re}\ \langle D^* D\varphi, \psi\rangle$$

where D^* is the formal L^2 adjoint of D. So the field equation is

$$D^* D\varphi = 0$$

But if this is satisfied, then

$$0 = \int_M \langle D^* D\varphi, \phi\rangle = \int_M |D\varphi|^2 \implies D\varphi = 0$$

assuming $\partial M = \varnothing$ (or $\varphi \to 0$ at ∞ quickly enough). Therefore all critical points of Φ are absolute minima.

Example 1.49. As a special case of the above, take

$$D = \mathrm{d}^* + \mathrm{d} : \Omega^p_M \to \Omega^{p-1}_M \oplus \Omega^{p+1}_M$$
$$\text{or} \quad D = \mathrm{d}^* + \mathrm{d} : \Gamma\left(\textstyle\bigwedge^p T^*M\right) \to \Gamma\left(\textstyle\bigwedge^{p-1} T^*M \oplus \textstyle\bigwedge^{p+1} T^*M\right)$$

Then

$$\Phi(\omega) = \frac{1}{2}\int_M |D\omega|^2 = \frac{1}{2}\int_M |\mathrm{d}\omega + \mathrm{d}^*\omega|^2 = \frac{1}{2}\int_M |\mathrm{d}\omega|^2 + |\mathrm{d}^*\omega|^2$$

The field equations give an absolute minimum

$$\begin{cases}\mathrm{d}\omega = 0\\ \mathrm{d}^*\omega = 0\end{cases} \iff \begin{cases}\mathrm{d}\omega = 0\\ \mathrm{d}(\star\omega) = 0\end{cases} \qquad \text{since } \mathrm{d}^* = \pm \star \mathrm{d}\star$$

1.9. **Hodge-de Rham Theory and Harmonic Forms.**

Definition 1.50. The Hodge laplacian is the operator on p-forms: $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$. Again, recall that d^* is the L^2 adjoint of d.

Definition 1.51. A p-form ω is called harmonic if $\mathrm{d}\omega = \mathrm{d}^*\omega = 0$, or equivalently $\Delta\omega = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\omega = 0$.

Note (to self). Here is a small proof. The forward direction is obvious. Assume $\Delta\omega = 0$. Then

$$0 = \langle \Delta\omega, \omega\rangle_{L^2} = \int \langle \mathrm{d}\mathrm{d}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle = \int |\mathrm{d}^*\omega|^2 + |\mathrm{d}\omega|^2 \geq 0$$

implies both $|\mathrm{d}^*\omega|$ and $|\mathrm{d}\omega|$ are identically 0 everywhere.

Theorem 1.52 (Hodge-de Rham)**.** *Let (M, g) be a compact Riemannian manifold. Then there is an L^2-orthogonal decomposition*

$$\Omega^p_M = \mathscr{H}^p \oplus \mathrm{d}\Omega^{p-1}_M \oplus \mathrm{d}^*\Omega^{p+1}_M \tag{13}$$

where $\mathscr{H}^p = \{$harmonic p-forms$\}$ is finite-dimensional.

Proof. The existence is by some analysis. The spaces are perpendicular since

$$\int_M \langle \mathrm{d}\alpha, \mathrm{d}^*\beta\rangle = \int_M \langle \underbrace{\mathrm{d}\mathrm{d}\alpha}_{=0}, \beta\rangle = 0$$
$$\int_M \langle h, \mathrm{d}\alpha\rangle = \int_M \langle \underbrace{\mathrm{d}^* h}_{=0}, \alpha\rangle = 0$$
$$\int_M \langle h, \mathrm{d}^*\beta\rangle = \int_M \langle \underbrace{\mathrm{d}h}_{=0}, \beta\rangle = 0$$

□

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Corollary 1.53. *Each de Rham cohomology class has a unique harmonic representative. Thus*

$$\mathscr{H}_g^p \cong H^p_{\mathrm{dR}}(M;\mathbb{R}) \tag{14}$$

Proof. Write the representative of $[\omega] \in H^p(M)$ as

$$\omega = h + \mathrm{d}\alpha + \mathrm{d}^*\beta.$$

But

$$0 = \mathrm{d}\omega = \mathrm{d}h + \mathrm{d}\mathrm{d}\alpha + \mathrm{d}\mathrm{d}^*\beta = \mathrm{d}\mathrm{d}^*\beta$$
$$0 = \langle \mathrm{d}\mathrm{d}^*\beta, \beta\rangle = \int_M |\mathrm{d}^*\beta|^2$$
$$\implies \mathrm{d}^*\beta = 0$$

So $\omega = h + \mathrm{d}\alpha \implies [h] = [\omega - \mathrm{d}\alpha] = [\omega]$.
Now if h and h' are two harmonic forms representing $[\omega]$, then $[h - h'] = 0$, i.e. $h - h' = \mathrm{d}\gamma$ for some $\gamma \in \Omega_M^{p-1}$. But then by decomposition (13), $h - h' = 0$. □

Generalization:

Recall a connection on $E \to M$ gives

$$\cdots \to \Omega^{p-1}(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^p(E) \xrightarrow{\mathrm{d}^\nabla} \Omega^{p+1}(E) \to \cdots$$

This is not a complex ($\mathrm{d}^\nabla \mathrm{d}^\nabla \neq 0$), but we can still define harmonic p-forms

$$\mathscr{H}^p_{\nabla,g}(E) := \{\omega \in \Omega^p(E) \ : \ \mathrm{d}^\nabla\omega = 0 \text{ and } (\mathrm{d}^\nabla)^*\omega = 0\}$$

and cohomology

$$H^p(M, E) := \frac{\ker \mathrm{d}^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)}{\operatorname{im} \mathrm{d}^\nabla : \Omega^{p-1}(E) \to \Omega^p(E)}$$

Then we have the generalized version of the Hodge Theorem:

$$\mathscr{H}^p_{\nabla,g}(E) \cong H^p(M, E)$$

but this is not topological, it depends on ∇. However, we still have:

Theorem 1.54. *If (M, g) is closed, then $\mathscr{H}^p_{\nabla,g}(E)$ is finite-dimensional for all ∇ and g, and the euler characteristic*

$$\chi(M, E) = \sum_{0}^{\dim M} (-1)^i \dim H^p(M, E)$$

depends only on M, E.

To calculate $\chi(M, E)$ use an "index theorem".

Prototype: For a 2-manifold M

$$\chi(M, E) = \dim_{\mathbb{C}} H^0(M, E) - \dim_{\mathbb{C}} H^{0,1}(M, E)$$

1.10. Riemann-Roch Theorem.

For a rank k vector bundle over a genus g 2-manifold,

$$\chi(M, E) = \underbrace{c_1(E)[M]}_{\in \mathbf{Z}} + k(1 - g)$$

In higher dimensions, the Hirzebruch index theorem and the Atiyah-Singer index theorem tell us that $\chi(M, E)$ = a specific polynomial P in: chern classes and Pontryagin classes of M, and chern classes of E. This implies that $P \in H^*(M)$ and $\chi(M, E) = P[M]$.

Remark. For 3-dimensions, all chern classes vanish.

2. Principal Bundles

In general, gauge theories are built from a Lie group G and a representation ρ of G. The roadmap is:

$$\text{Lie group } G \rightsquigarrow \text{Principal } G\text{-bundle} \overset{\rho}{\rightsquigarrow} \text{Gauge Theory}$$

2.1. **Beginning with principal bundles.**

Definition 2.1. Fix a manifold M and Lie group G. A principal $G-$bundle over M is a smooth map $P \xrightarrow{\pi} M$ from a manifold P with an action of G on P, such that

(i) G acts freely on P by "right multiplication":
$$p \mapsto pg^{-1} = R_g(p)$$
Note $R_{gh} = R_g R_h$. (Free action: For each $g \in G$, if there is a $p \in P$ with $pg^{-1} = p$, then g is the identity.)

(ii) $M = P/G$

(iii) P is locally trivial: For all $x \in M$, there is a neighborhood U and a fiber-preserving G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times G$. ($G$-equivariant: $\varphi(pg^{-1}) = \varphi(p) \cdot g^{-1}$.)

Note. A principal bundle can also be described as a fiber bundle with structure group and fiber both "equal" to G (see Nash & Sen). One should still add the locally trivial property to this, to see that the structure group G acts on the fiber G naturally by multiplication (on the right, by the inverse if needed). There are other bundles with fiber equal to G, but which are not principal bundles, see $C_G(P)$ below.

Example 2.2. The trivial $G-$bundle $M \times G \to M$.

Example 2.3 (Hopf fibration). $S^1 = U(1)$ acts on the unit sphere $S^{2n+1} \subset \mathbb{C}$ by coordinate multiplication $e^{i\theta}(z^1, \cdots, z^{n+1}) = (e^{i\theta}z^1, \cdots, e^{i\theta}z^{n+1})$. This action is free, with disjoint orbits all diffeomorphic to S^1. Then

$$\begin{array}{c} P = S^{2n+1} \\ \downarrow \\ M = \mathbb{C}P^n \end{array}$$

is a principal S^1-bundle. (e.g. $S^3 \to S^2 = \mathbb{C}P^1$)

Example 2.4. $G = SU(2) = \{\text{unit quaternions}\} \cong S^3$ (right) acts on the unit sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$. Thus

$$\begin{array}{c} P = S^{4n+3} \\ \downarrow \\ M = \mathbb{H}P^n \end{array}$$

is a principal $SU(2)$-bundle. (e.g. $S^7 \to S^4$ is a non-trivial $SU(2)$-bundle over S^4)

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Main example of a principal bundle

For a Riemannian manifold (M^n, g), we have the notion of an ON (orthonormal) frame at $p \in M$, $e = \{e_1, \cdots, e_n\}$, which gives an isometry $T_pM \cong \mathbb{R}^n$. Note any two such frames at p are related by some unique $g \in O(n)$.

Definition 2.5. The O(n) frame bundle of (M, g) is

$$\mathrm{Fr}(M) = \{(p, e) \ : \ p \in M, e = \{e_1, \cdots, e_n\} \text{ ON frame at } p\}$$

with the obvious projection to M $\pi(p, e) = p$. The group $G = O(n)$ acts by $(p, e).g^{-1} = (p, ge)$

Lemma 2.6. $\mathrm{Fr}(M)$ *is a principal* $O(n)$ *bundle.*

Proof. The above action is

- free: $(p, ge) = (p, e) \implies g = \mathrm{id}$
- fiber-preserving
- transitive on each fiber: for each $(p, e), (p, e')$, there is a $g \in O(n)$ such that $ge = e'$
- locally trivial: Given $p \in M$, take a coordinate neighborhood $\{x^i\}$ on U. Then $\left\{v_i = \frac{\partial}{\partial x^i}\right\}$ are linearly independent vector fields. By Gram-Schmidt, we can get vector fields $e = \{e_1, \cdots, e_n\}$, ON at each $q \in U$. Then set

$$\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & \mathrm{Fr}(M)\big|_U \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

This is an equivariant isomorphism, making $\mathrm{Fr}(M)$ a manifold.

□

Example 2.7. If (M^n, g) is oriented, we can consider positively oriented ON frames $e = \{e_1, \cdots, e_n\}$. We have $\mathrm{Fr}^+(M) \subset \mathrm{Fr}(M)$ a subbundle over M, and a principal $SO(n)-$bundle.

Example 2.8. For a manifold M^n (with no metric), a (general) frame at $p \in M$ is a linearly independent set $e = \{e_1, \cdots, e_n\} \iff T_pM \cong R^n$. This leads to a GL$(n)$ frame bundle $\widetilde{\mathrm{Fr}}(M) \to M$

Remark. Adding geometric structure allows us to "reduce" the frame bundle

$$\begin{array}{ccccc} \mathrm{Fr}^+(M) & \overset{\text{orient}}{\hookrightarrow} & \mathrm{Fr}(M) & \overset{\text{metric}}{\hookrightarrow} & \widetilde{\mathrm{Fr}}(M) \\ & \searrow & \searrow & & \downarrow \\ & & & & M \end{array}$$

Example 2.9. For a rank k complex vector bundle $E \to M$ with hermitian metric $\langle \ , \ \rangle$, we have the associated frame bundle is a principal $U(k)$-bundle

$$\mathrm{Fr}(E) := \{(p, e) \ : \ p \in M, e = (e_1, \cdots, e_n), \text{ ON basis of } E_p, \text{ i.e. } E_p \overset{e}{\cong} \mathbb{C}^k\}$$

Remark. A local framing of E is a section of $\mathrm{Fr}(E)$ over some $U \subset M$. We can always construct these as in the proof of Lemma 2.6.
Note that a principal G-bundle admits a global section if and only if P is trivial.

Example 2.10. Let (M^4, g) be a space-time manifold:

$$g \sim \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = g_0$$

A frame at p is a basis $\{e_0, e_1, e_2, e_3\}$ of T_pM such that

$$\begin{cases} \langle e_i, e_j \rangle = 0 \\ |e_0|^2 = -1 \\ |e_j|^2 = 1 \end{cases} \iff \text{an isometry } T_pM \cong \mathbb{R}^{1,3} = (\mathbb{R}^4, g_0)$$

2.2. **Associated Vector Bundles.**

Fix a Lie group G. Consider a representation ρ of G on a vector space V, i.e.

$$\rho : G \to \mathrm{End}(V) \quad \text{such that } \rho(gh) = \rho(g)\rho(h)$$

Definition 2.11. Given a principal G-bundle $P \to M$ and a representation ρ of G on a vector space V. The associated vector bundle is a bundle $E \to M$ defined by

$$E = P \times_\rho V = \frac{\big\{(p, v) \in P \times V\big\}}{(p, v) \sim (pg^{-1}, \rho(g)v)}$$

Exercise 2.12. Show

$$\Gamma(E) = \{G\text{-equivariant maps } P \to V\} \quad \big(:= \{\varphi : P \to V \ : \ \varphi(pg^{-1}) = \rho(g)\varphi(p)\} \ \big)$$

Proof. We will write $[p, v]$ to denote a point in E (the equivalence class of (p, v)).

Let $\varphi : P \to V$ be equivariant. Define a map $\sigma : M \to E$ by

$$\sigma(x) = [p, \varphi(p)] \tag{15}$$

where p is any point in the fiber over x of $P \to M$. This is well defined since, if we use a different point in the fiber pg^{-1}, then

$$[pg^{-1}, \varphi(pg^{-1})] = [pg^{-1}, \rho(g)\varphi(p)] = [p, \varphi(p)].$$

The projection of $E \to M$ is defined by using the map $P \xrightarrow{\pi} M$, i.e. $[(p, v)] \mapsto \pi(p) = x$, so our σ is a section of E, by our choice of p over x.

Now let $\sigma \in \Gamma(E)$. Then define $\varphi : P \to V$ to be the map which makes (15) true. That is, to define $\varphi(p)$, let $x = \pi(p)$. Then $\sigma(x) = [q, w]$ for some q, w. Because the action of G is free and transitive on the fiber over x, there is a unique g such that $qg^{-1} = p$. Then $\sigma(x) = [q, w] = [p, \rho(g)w]$ is of the form of (15). This is G-equivariant by

$$[pg^{-1}, \varphi(pg^{-1})] = \sigma(x) = [p, \varphi(p)] = [pg^{-1}, \rho(g)\varphi(p)]$$

which implies $\varphi(pg^{-1}) = \rho(g)\varphi(p)$. □

Exercise 2.13. For a complex rank k vector bundle $E \to M$ with $\langle\ ,\ \rangle$. Show that E is associated to the standard representation of its frame bundle, i.e. that

$$E = \mathrm{Fr}(E) \times_\rho \mathbb{C}^k \qquad \text{where } \rho : U(k) \to \mathrm{End}(\mathbb{C}^k) \text{ is the "standard representation"}$$

Example 2.14. Fix (M^n, g), which gives us $\mathrm{Fr}(M)$, a principal $O(n)$-bundle.

(1) With the standard representation $\rho : O(n) \to \mathrm{End}(\mathbb{R}^n)$, we have
$$\mathrm{Fr}(M) \times_\rho \mathbb{R}^n = TM$$
(2) With the dual representation $\rho^* : O(n) \to \mathrm{End}((\mathbb{R}^n)^*)$ given by
$$(\rho^*(g)\alpha)v = \alpha(\rho(g)v) \quad \text{for } \alpha \in (\mathbb{R}^n)^*,\ v \in \mathbb{R}^n,$$
we have
$$\mathrm{Fr}(M) \times_{\rho^*} (\mathbb{R}^n)^* = T^*M$$
(3) With the representation $\rho_k^* : O(n) \to \mathrm{End}\left(\bigwedge^k (\mathbb{R}^n)^*\right)$ given by
$$\rho_k^*(g)(\alpha_1 \wedge \cdots \wedge \alpha_k) = \rho^*(g)\alpha_1 \wedge \cdots \wedge \rho^*(g)\alpha_k,$$
we have
$$\mathrm{Fr}(M) \times_{\rho_k^*} \bigwedge^k (\mathbb{R}^n)^* = \bigwedge^k T^*M$$
Hence a differential k-form corresponds to an equivariant map $\mathrm{Fr}(M) \to \bigwedge^k (\mathbb{R}^n)^*$, (by exercise 2.12)

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So given a principal G bundle P, and a representation ρ of G on V, we get an associated vector bundle $E = P \times_\rho V$. More generally,

Definition 2.15. If G acts on a space X, $\alpha : G \times X \to X$, we get an associated fiber bundle
$$P \times_\alpha X = \frac{\{(p, x) \in P \times X\}}{(p, x) \sim (pg^{-1}, \alpha(g).x)}$$
with fiber homeomorphic to X. The transition functions will be α applied to the transition functions for P.

Proposition 2.16. *The construction above is a smooth fiber bundle (as long as α is smooth), with transition functions given by applying α to the transition functions for P.*

Proof. (Check: proof) □

The following constructions are important in gauge theory:

Definition 2.17. The Lie group G acts on G itself by conjugation C:

$$C : G \times G \to G \quad \text{by } (g,h) \mapsto C_g(h) = ghg^{-1} \tag{16}$$

The associated bundle, called the Adjoint bundle, $C_G(P) = P \times_C G$ is a bundle whose fibers are copies of G. However $C_G(P)$ is not a *principal* G-bundle (if P is nontrivial). We have (see exercise 2.12)

$$\Gamma(C_G(P)) = \{s : P \to G \;:\; s(pg^{-1}) = C_g(s(p)) = gs(p)g^{-1}\} \tag{17}$$

The map $s(p) \equiv e$ is a well-defined section, so $\Gamma(\mathrm{Ad}(P)) \neq 0$. Note also that we can multiply sections; $\Gamma(\mathrm{Ad}(P))$ is a group.

Definition 2.18. The gauge group $\mathcal{G}$ of P is the group of all bundle automorphisms covering the identity on M, and such that $\varphi(pg^{-1}) = \varphi(p)g^{-1}$ for all $p \in P, g \in G$.

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Exercise 2.19. The gauge group $\mathcal{G} \cong \Gamma(\mathrm{Ad}(P))$ (as groups) by $\varphi_s \leftrightarrow s$, where $\varphi_s(p) = p \cdot s(p)^{-1}$

Proof. If $s : P \to G$ is an equivariant map as in (17), then define $\varphi_s(p) = p \cdot s(p)^{-1}$ as above. Then φ_s is a bundle map, since it is defined using the G action on P. It is invertible since we can multiply by $s(p)$ everywhere. Then we check

$$\varphi_s(pg^{-1}) = pg^{-1}s(pg^{-1})^{-1} = pg^{-1}(gs(p)g^{-1})^{-1} = pg^{-1}gs(p)^{-1}g^{-1} = ps(p)^{-1}g^{-1} = \varphi_s(p)g^{-1}.$$

On the other hand, if we have $\varphi : P \to P$, then $\varphi(p)$ lies in the same fiber as p, so there is a unique $s(p) \in G$ such that $\varphi(p) = p \cdot s(p)^{-1}$. This defines a map $s : P \to G$. Using the local triviality to work in $U \times G$, we can see that φ is smooth implies that s is smooth. Now we figure out what $s(pg^{-1})$ is:

$$pg^{-1}s(pg^{-1})^{-1} = \varphi(pg^{-1}) = \varphi(p)g^{-1} = ps(p)^{-1}g^{-1} = pg^{-1}(gs(p)g^{-1})^{-1}$$

Since the action is free, we have $s(pg^{-1}) = gs(p)g^{-1}$.

(Check: explain why the constructed maps φ_s and s are smooth.)

We also have

$$(\varphi_t\psi_s)(p) = \varphi_s(\varphi_t(p)) = \varphi_s(pt(p)^{-1}) = \varphi_s(p)t(p)^{-1} = ps^{-1}t^{-1} = p(ts)^{-1}$$

i.e. the assignment is a group homomorphism. □

Note. $\mathcal{G}$ is not $\mathrm{Map}(M, G)$, unless P is the trivial bundle $P = M \times G$.

Definition 2.20. For each $g \in G$, $C_g : G \to G$ preserves the identity e in G. So we take its derivative

$$\begin{aligned} (C_g)_{*,e} &: T_eG \to T_eG \\ = \mathrm{Ad}(g) &: \quad \mathfrak{g} \to \mathfrak{g} \end{aligned}$$

This is the Adjoint representation $G \to \mathrm{GL}(\mathfrak{g})$ (invertible elements of $\mathrm{End}(\mathfrak{g})$). The associated vector bundle is called the adjoint bundle

$$\mathrm{ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$$

Note (about ad). Say $G \leq \mathrm{GL}_n$ is some matrix Lie group. Then for $g, h \in G$, conjugation is

$$C_g(h) = ghg^{-1} \tag{18}$$

where this is explicitly matrix multiplication. The tangent space to the identity $\mathfrak{g}$ is a subspace of $T_I\mathrm{GL}_n = M_{n\times n}$. So for a vector $Y \in \mathfrak{g}$, we can apply the derivative of conjugation at I, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. Explicitly, differentiating (18) with respect to h gives

$$\mathrm{Ad}_g(Y) := (C_g)_{*,I}(Y) = gYg^{-1} \tag{19}$$

where this last expression is matrix multiplication again, since Y is an $n \times n$ matrix.

Now the assignment $g \mapsto \mathrm{Ad}_g$ is actually a Lie group homomorphism

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) = \{\mathfrak{g} \to \mathfrak{g},\ \text{invertible linear}\}.$$

The identity $I \in G$ is sent to $(X \mapsto IXI^{-1} = X)$, the identity of $\mathrm{GL}(\mathfrak{g})$.
Taking the derivative of this new map,

$$\begin{aligned} \mathrm{ad} := (\mathrm{Ad})_{*,I} : \mathfrak{g} &\to \mathrm{Lie}(\mathrm{GL}(\mathfrak{g})) = \mathrm{End}(\mathfrak{g}) = \{\mathfrak{g} \to \mathfrak{g},\ \text{linear}\} \\ X &\mapsto \mathrm{ad}_X \end{aligned}$$

Explicitly, we can differentiate (19) for a path $g(t) \in G$ with $g(0) = I$ and $g'(0) = X \in \mathfrak{g}$.

$$\begin{aligned} \mathrm{ad}_X(Y) &= g'Yg^{-1} + gY(g^{-1})' \\ &= XY - YX = [X, Y]. \end{aligned}$$

Recall that the Lie algebra structure on $\mathrm{End}(V)$ is defined using this explicit bracket. It is in fact true in general (for abstract Lie groups/algebras) that $\mathrm{ad}_X(Y) = [X, Y]$, using the Lie bracket on $\mathfrak{g}$.

2.3. **Connections on principal bundles.**

The set up is a principal bundle $P \xrightarrow{\pi} M$.

The idea of a connection is one of "infinitesimal parallel transport". One needs to figure out how to drag elements of G across fibers in P, in a way that is "parallel" to the direction of M.

For a point in P, locally (x, g), we can move in the fiber over x, changing g only. This is "vertical" movement, which is a notion invariant of trivialization. If we try to move "horizontally" (change x but keep g constant), then this will not be invariant: Under different trivializations, the same motion may not keep g constant. To define a connection, we will choose a direction to be called horizontal. We will start by using the fiberwise action of G to move vertically.

Consider the G-action

$$\begin{aligned} G \times P &\to P \\ (g, p) &\mapsto pg^{-1} = R_g(p). \end{aligned}$$

The derivative gives a linear map for each $g \in G$

$$(R_g)_* : T_pP \to T_{pg^{-1}}P$$

and for each $p \in P$, a linear injection

$$\begin{aligned} i_p &: T_eG \to T_pP \\ \text{or} \quad i_p &: \mathfrak{g} \to T_pP \quad \text{with } \pi_* i_p = 0 \end{aligned}$$

i.e. vertical movement.

Definition 2.21. The vertical subspace of T_pP is

$$\begin{aligned} V_p = \ker(\mathrm{d}\pi)_p &= \text{image } i_p \subset T_pP \\ &= \text{tangent space of } G\text{-orbit through } p. \end{aligned}$$

Each $A \in \mathfrak{g}$ has an associated fundamental (vertical) vector field on P.

$$(A^*)(p) = i_p(A)$$

(Check: I think I remember that fundamental vector fields are (related to) left-invariant vector fields - look for some exercise in Tu.)

Definition 2.22. A connection on P is a choice of horizontal subbundle $H \subset TP$ which is

- horizontal: $H_p \oplus V_p = T_pP$, for each $p \in P$
- equivariant: $H_{pg^{-1}} = (R_g)_* H_p$

So with a connection defined, at each $p \in P$ we have projections

$$\begin{aligned} v &: T_pP \to V_p \\ h &: T_pP \to H_p \end{aligned}$$

The composition

$$T_pP \xrightarrow{v_p} V_p \xrightarrow{i_p^{-1}} \mathfrak{g}$$

is a $\mathfrak{g}$-valued 1-form on P.

Definition 2.23. A connection form is a $\mathfrak{g}$-valued 1-form ω on P such that

(1) $\omega(A^*) = A$ (i.e. $\omega(i_p(A)) = A$ for all $A \in \mathfrak{g}$)
(2) $(R_g)^*\omega = (\mathrm{Ad}\, g)\omega$ for all $g \in G$.

Lemma 2.24. *Definitions 2.22 and 2.23 are equivalent by* $H_p = \ker \omega_p$.

This should be interpreted as: ω is the horizontal projection (projection onto V following the lines of H).

Proof. (cited Kobayashi & Nomizu, vol.1 p.64) □

Exercise 2.25. Fix a connection ω. Show that another $\mathfrak{g}$-valued 1-form ω is a connection form if and only if $\omega' = \omega + \eta$ for some $\eta \in \Omega^1_G(P)$ where

$$\Omega^p_G(P) := \{\eta \in \Omega^p(P, \mathfrak{g}) \ : \ (R_g)^*\eta = (\mathrm{Ad}\, g)\eta, \ \text{ and } \ \eta(A^*) = 0 \quad \forall A \in \mathfrak{g}\}$$

Consequences

The choice of connection ω (or H_p) determines:

1. An exterior covariant derivative $\mathrm{D} : \Omega^p_G(P) \to \Omega^{p+1}_G(P)$ by

$$\mathrm{D}\eta(X_0, \cdots, X_p) = \mathrm{d}\eta(hX_0, \cdots, hX_p)$$

Note this is equal to 0 if any X_i is vertical.

2. The curvature of ω is the $\mathfrak{g}$-valued 2-form

$$\Omega = \mathrm{D}\omega \in \Omega^2_G(P)$$

Proposition 2.26.
(a) *For η, [Check: should have some horizontal or basic condition]*

$$\mathrm{D}\eta = \mathrm{d}\eta + [\eta \wedge \omega]$$

$$\left(i.e.\ \mathrm{D}\eta(X,Y) = \mathrm{d}\eta + \frac{1}{2}\big([\eta(X), \omega(Y)] + [\omega(X), \eta(Y)]\big) \right)$$

(b) (Structure Equation)

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega$$

(c) *If X, Y are horizontal,*

$$\Omega(X,Y) = -\frac{1}{2}\omega\big([X,Y]\big)$$

Proof. (cited Kobayashi & Nomizu)
For (c), note: each vector field X on M has a horizontal lift $\widetilde{X}$ on P, the unique $\widetilde{X} \in H$ such that $\pi_*\widetilde{X} = X$. Then

$$\Omega(\widetilde{X}, \widetilde{Y}) = -\frac{1}{2}\omega\big([\widetilde{X}, \widetilde{Y}]\big) = -\frac{1}{2}v\big([\widetilde{X}, \widetilde{Y}]\big)$$

So, curvature is a measure of the failure of $H \subset TP$ to be a distribution (i.e. closed under [,]). □

3. The differential Bianchi identity

$$\mathrm{D}\Omega = 0 \quad \in \Omega^3_G(P)$$

4. For each path $\gamma : [a,b] \to M$, we get a horizontal lift $\widetilde{\gamma} : [a,b] \to P$ to $p \in P$, with $\pi(p) = \gamma(a)$, such that

(a) $\pi\widetilde{\gamma} = \gamma$, (lifts γ)
(b) $\widetilde{\gamma}' \in H$ for all t, (moves horizontally)

Lemma 2.27. *For each such γ, there exists a unique horizontal lift $\widetilde{\gamma}$.*

Proof. First choose some lift $\sigma(t)$ of γ, so $\pi\sigma = \gamma$. Now we write the desired lift as

$$\widetilde{\gamma}(t) = \sigma(t)g(t)$$

Then the horizontal condition $\widetilde{\gamma}' \in H = \ker\omega$ iff

$$g' = -\omega(\sigma')g$$

This is an ODE, so there exists a unique solution $g(t)$ for $\widetilde{\gamma}(a) = p$. □

A horizontal lift is completely analogous to parallel transport.

5. We get an induced parallel transport on every associated vector bundle $E = P \times_\rho V$:

Fix a starting point $q \in M$ and an initial vector $\xi(0) \in E_q$, which we write as (p, v). Given a path $\gamma(t)$ in M starting at q, define

$$\xi(t) = (\widetilde{\gamma}(t), v)$$

a section of E over $\gamma(t)$. Then the parallel transport map is $P^\omega_{\gamma(t)} : E_{\gamma(a)} \to E_{\gamma(t)}$ defined by

$$P^\omega_{\gamma(t)}\xi(0) = \xi(t)$$

If we choose a different representation $(p, v) \mapsto (pg^{-1}, \rho(g)v)$ we have $\xi \mapsto \xi g^{-1}$ [Check: prove this].

6. From parallel transport, we get a covariant derivative on E by

$$(\nabla_X \xi)(q) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(P^\omega_{\gamma(t)}\right)^{-1}\xi(t)\right)\Big|_{t=0}$$

for $\xi \in \Gamma(E), X \in T_qM$, where $\gamma(t)$ is a curve in M starting at q with initial velocity X.

Exercise 2.28. This is independent of γ, linear in ξ, C^∞-linear in X. Thus ∇ is a connection on E.

Going backwards

Given a hermitian vector bundle $E \to M$, we let $P = \mathrm{Fr}(E)$.

Lemma 2.29. *A connection on E is equivalent to a horizontal subspace (a connection on P).*

Proof. If we have a connection on E and $p = \{e_1, \cdots, e_k\} \in \mathrm{Fr}(E)$ an ON frame at $q = \pi(p) \in M$, for each vector $X \in T_qM$, choose a path in M continuing X. Parallel transport defines ON sections $\{e_1(t), \cdots, e_k(t)\}$ along γ. This gives us a path $\widetilde{p}(t) = \{e_i\}$ in $P = \mathrm{Fr}(E)$. Then the vector $V = \widetilde{p}'(0)$ is what we should call horizontal. Set

$$H_p = \{V \in T_pP \quad \text{obtained in this way}\}$$

Then we check that $H_p \pitchfork V_p$.

Also $\widetilde{p}(t)g^{-1}$ is another such path,

$$\implies H_{pg^{-1}} = (R_g)_* H_p$$

□

3. Gauge Theory - a new viewpoint

3.1. Potentials.

Definition 3.1. A (classical) gauge theory consists of:

(1) A Lie group G and some representations $\rho_1, \cdots, \rho_l$ of G on $V_1, \cdots, V_l$ respectively
(2) A manifold M with a (pseudo-) Riemannian metric g
(3) A principal G-bundle $P \xrightarrow{\pi} M$
(4) An action functional ($\mathcal{G}$-invariant)

$$\Phi(\nabla, \varphi_1, \cdots, \varphi_l) = \frac{1}{2}\int_M \mathscr{L}\left(F^\nabla, \varphi_1, \cdots, \varphi_l, \nabla\varphi_1, \cdots, \nabla\varphi_l\right) \mathrm{dvol}_g$$

where $\varphi_i \in \Gamma(E_i), E_i = P \times_{\rho_i} V_i$ are sections, and ∇ is a connection on P.
The critical points of $\Phi : \mathcal{A} \times \Gamma(E_1) \times \cdots \times \Gamma(E_l) \to \mathbb{R}$ are solutions of the Euler-Lagrange equations (the "field equations").

Example 3.2. Let $G = U(k)$ with the standard representation $\rho : U(k) \to \mathrm{GL}(\mathbb{C}^k)$, $\implies E = P \times_\rho \mathbb{C}^k$, the action

$$\Phi(\nabla, \varphi) = \frac{1}{2}\int_M \left|F^\nabla\right|^2 + |\nabla\varphi|^2 + m|\varphi|^2$$

(which has a mass term, $m \in \mathbb{R}$). The field equations are

$$\begin{cases} (\mathrm{d}^\nabla)^* F = j_\varphi & \text{current} \\ (\nabla^*\nabla + m)\varphi = 0 \end{cases}$$

where $j_\varphi(\eta) = \mathrm{Re}\,\langle \nabla\varphi, \rho(\eta)\varphi\rangle$.
Note the second equation gives us that either $m < 0$ or $\varphi \equiv 0$.

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Example 3.3. Let's play with the potential (the action) in an effort to move the minimum away from $\varphi = 0$.

$$\Phi(\nabla, \varphi) = \frac{1}{2}\int_M \left|F^\nabla\right|^2 + |\nabla\varphi|^2 + \underbrace{\lambda\left(|\varphi|^2 - m\right)^2}_{\text{Higgs potential}}$$

with $\lambda, m > 0$. Now we have minima at $|\varphi|^2 = m > 0$. The field equations are

$$\begin{cases} (\mathrm{d}^\nabla)^* F = j_\varphi \\ \nabla^*\nabla\varphi = -4\lambda\left(|\varphi|^2 - m\right)\varphi \end{cases}$$

Example 3.4. Let $G = U(k)$. Take ρ_1 the standard representation of $U(k)$ on C^k, and ρ_2 the adjoint representation of $U(k)$ on its Lie algebra $\mathfrak{u}(k)$, described by

$$\rho_2(g) : X \mapsto gXg^{-1}$$

Each $X \in \mathfrak{u}(k)$ gives a linear transformation $\mathbb{C}^k \to \mathbb{C}^k$, and we get a G-equivariant linear map

$$\mathfrak{u}(k) \times \mathbb{C}^k \to \mathbb{C}^k$$

$$(X, v) \mapsto X.v \tag{20}$$
$$(\rho_2(g)X.\rho_1(g)v) = \rho_1(g)(X.v)$$

Exercise 3.5. A G-equivariant linear map $\alpha : V_1 \to V_2$ between representations $\rho_i : G \to \mathrm{GL}(V_i)$ induces a vector bundle map $E \to F$ for $E = P \times_{\rho_1} V_1$, $F = P \times_{\rho_2} V_2$

Hence the map (20) induces a bundle map $\mathrm{ad}(P) \times E \xrightarrow{\alpha} E$

Now take the potential

$$\Phi(\nabla, \varphi, \psi) = \frac{1}{2}\int_M \left|F^\nabla\right|^2 + |\nabla\varphi|^2 + |\nabla\psi|^2 + \underbrace{\langle \varphi, \alpha(\psi, \varphi)\rangle}_{\text{interaction term}}$$

where $\varphi \in \Gamma(E), \psi \in \Gamma(\mathrm{ad}(P))$. The E-L equations form a coupled non-linear elliptic system.

$$\begin{cases} (\mathrm{d}^\nabla)^* F^\nabla = j_\varphi + j_\psi \\ \nabla^*\nabla\varphi = -\mathrm{Re}\,\langle \cdot, \alpha(\psi, \varphi)\rangle \\ \nabla^*\nabla\psi = -\langle \varphi, \alpha(\cdot, \varphi)\rangle \end{cases}$$

3.2. **Remarks on abelian gauge theories I. Gauge group.**

Let $P \xrightarrow{\pi} M$ be a principal $G = U(1)$-bundle. Since G is abelian, the gauge group is (recall exercise 2.19)

$$\begin{aligned} \mathcal{G} = \Gamma(\mathrm{Ad}\,P) &= \Gamma(P \times_{\mathrm{Ad}} G) \\ &= \{\varphi : P \to G \ : \ \varphi(pg^{-1}) = g\varphi(p)g^{-1}\} \\ &= \{\varphi : P \to G \ : \ \varphi(pg^{-1}) = \varphi(p)\ \forall g\}. \end{aligned}$$

i.e. φ is constant on the fibers of π, and descends to M:

$$\mathcal{G} = \mathrm{Maps}(M, S^1)$$

which is a group by the multiplication in S^1, so $\mathcal{G}$ is abelian. This contains the subgroup

$$\mathcal{G}_1 = \{e^{if(x)} \ : \ f \in C^\infty(M)\} \subset \mathcal{G}$$

of maps homotopic to the identity (by $e^{itf(x)}$, $t \to 0$).

Lemma 3.6. *There is a short exact sequence of groups*

$$1 \to \mathcal{G}_1 \xrightarrow{i} \mathcal{G} \xrightarrow{j} H^1(M, \boldsymbol{Z}) \to 0$$

Proof. The inclusion i is injective.

We want to show exactness at $\mathcal{G}$. For $\varphi \in \mathcal{G} = \{M \to S^1\}$, define

$$j(\varphi) = [\varphi^*\mathrm{d}\theta] \in H^1_{\mathrm{dR}}(M)$$

Note $\mathrm{d}\varphi^*\mathrm{d}\theta = \varphi\mathrm{d}\mathrm{d}\theta = 0$, so $\varphi^*\mathrm{d}\theta$ is closed and defines a cohomology class. For any closed loop $\gamma \in M$,

$$\int_\gamma \varphi^*\mathrm{d}\theta = \int_{\varphi(\gamma)} \mathrm{d}\theta = \text{winding number} \in \mathbf{Z}$$

Hence $[\varphi^*\mathrm{d}\theta] \in H^1(M, \mathbf{Z}) \subset H^1_{\mathrm{dR}}(M, \mathbb{R})$.

If $\varphi(x) = e^{if(x)}$, then $\varphi^*\mathrm{d}\theta = \mathrm{d}f$ is exact, so $[\varphi^*\mathrm{d}\theta] = 0$. This proves $\mathrm{im}\, i \subset \ker j$.

(Just to make sure) let's see that $\varphi^*\mathrm{d}\theta = \mathrm{d}f$. The function $\theta : e^{i\psi} \mapsto \psi$ is a locally defined function with differential $\mathrm{d}\theta$. So locally we can compute using the also locally define $\theta \circ \varphi$

$$\varphi^*\mathrm{d}\theta = \mathrm{d}(\theta \circ \varphi) = \mathrm{d}\left(\theta(e^{if})\right) = \mathrm{d}f.$$

Finally, let $[\alpha] \in H^1(M, \mathbf{Z})$ be represented by a closed 1-form α. Fix $p \in M$ and define

$$f(x) = 2\pi \int_\gamma \alpha$$

for γ a path from p to x. Since $\mathrm{d}\alpha = 0$, this definition is independent of homotopic paths γ. In the case that α is exact, then f is globally well defined and smooth, since the integral of α around any loop is 0. We have $\mathrm{d}f = 2\pi\alpha$, and if we define the map

$$g(x) = e^{if(x)},$$

then this proves $\ker j \subset \mathrm{im}\, i$. Continuing with this if α is not exact, note that $g(x)$ is still a smooth function $M \to S^1$, since if a loop η represents a nontrivial homotopy class, then $\int_\eta \alpha$ is an integer, so $e^{2\pi i \int_\gamma \alpha}$ is well defined, and this proves that j is surjective. Note that $g^{-1}\mathrm{d}g = 2\pi i\alpha$. □

3.3. **Remarks on abelian gauge theories II. Global picture.**

We have the space of connections $\mathcal{A}$, and the quotient (the orbit space) $\mathcal{B} = \mathcal{A}/\mathcal{G}$. The gauge orbit through $\nabla \in \mathcal{A}$ is

$$\mathcal{O}^\nabla = \{\nabla' = g \circ \nabla \circ g^{-1} = \nabla - (dg)g^{-1} \;:\; g \in \mathcal{G}\}$$

Note that the constant gauge transformations $g_c : M \to S^1, g_c(x) = c$, act trivially on $\mathcal{A}$. ($\nabla = g\nabla g^{-1} \iff g = g_c$.) We want to ignore this degree of freedom, so we can remove it by replacing $\mathcal{G}$: (one of the following)

- Use $\mathcal{G}/\{\text{constants } g_c\}$.
- If M is compact, fix $p \in M$ and use $\mathcal{G}_p = \{g \in \mathcal{G} \;:\; g(p) = \mathrm{Id}\}$, the "based gauge group".
- If M is not compact, use $\overline{\mathcal{G}} = \{g \in \mathcal{G} \;:\; g \to \mathrm{Id} \text{ at } \infty\}$.

Recall that $\mathcal{A}$ is an affine space.

Lemma 3.7. *For an abelian gauge theory,*
(a) *$\mathcal{G}$-orbits $\mathcal{O}^\nabla$ are affine subspaces of $\mathcal{A}$.*
(b) *There exists a slice $\mathcal{S}^\nabla$ through each $\nabla \in \mathcal{A}$ (see below).*

Proof.
(a) For $g = e^{if} \in \mathcal{G}_1$,

$$\mathcal{O}^\nabla = \{\nabla + i\mathrm{d}f : f \in C^\infty(M)\} = \nabla + (\text{image } \mathrm{d} : C^\infty \to \Omega^1_M)$$

is an affine subspace.
(b) For $\nabla \in \mathcal{A}$, set

$$\mathcal{S}^\nabla = \{\nabla + iA \;:\; A \in \Omega^1_M,\ \mathrm{d}^*A = 0\}.$$

(Recall that for a linear operator T, $\ker T^* \perp \mathrm{im}\, T$.) We claim that $\mathcal{S}^\nabla$ intersects each $\mathcal{G}_1$ orbit exactly once (i.e. a slice for $\mathcal{G}_1$).

Given $\nabla' = \nabla + iB$, we need to find g such that $g.(\nabla + iB) \in \mathcal{S}^\nabla$. So we need

$$\begin{aligned} 0 &= \mathrm{d}^*[g.(\nabla + iB) - \nabla] \\ &= \mathrm{d}^*[\nabla + g\mathrm{d}g^{-1} + iB - \nabla] \\ &= \mathrm{d}^*[g\mathrm{d}g^{-1} + iB] \end{aligned}$$

We use the Hodge theorem to write

$$B = h + \mathrm{d}f + \mathrm{d}^*\gamma$$

for h harmonic, $f \in C^\infty(M)$, $\gamma \in \Omega^2_M$. Set $g = e^{if} \in \mathcal{G}_1$. Then

$$\mathrm{d}^*(iB + g\mathrm{d}g^{-1}) = i[\underset{\bullet}{\mathrm{d}^*h} + \mathrm{d}^*\mathrm{d}f + \underset{\bullet}{\mathrm{d}^*\mathrm{d}^*\gamma} + \mathrm{d}^*((g\mathrm{d}g^{-1})] = i[\mathrm{d}^*\mathrm{d}f - \mathrm{d}^*(\mathrm{d}f)] = 0$$

where the marked terms are vanish since h is harmonic ($\mathrm{d}h = \mathrm{d}^*h = 0$) and $(\mathrm{d}^*)^2 = 0$ (for any function θ, $\langle \mathrm{d}^*\mathrm{d}^*\gamma, \theta\rangle_{L^2} = \langle \gamma, \mathrm{d}\mathrm{d}\theta\rangle_{L^2} = 0$). Finally $H^1(M, \mathbf{Z})$ acts on $\mathcal{S}^\nabla$ by $\nabla + B \mapsto \nabla + B + \alpha$, for α harmonic ($\mathrm{d}^*\alpha = 0$). Thus

$$\mathcal{B} = \mathcal{S}^\nabla / H^1(M, \mathbf{Z})$$

□

3.4. **Self-Duality.**

Definition 3.8. Let V be an n-dimensional vector space with a metric $\langle\ ,\ \rangle$ and orientation $(e_1, \cdots, e_n)$ (and thus volume element $\mathrm{d}v = e_1 \wedge \cdots, \wedge e_n$). The Hodge star operator

$$\star : \bigwedge^p V^* \to \bigwedge^{n-p} V^*$$

is defined by the following: For all $\omega, \eta \in \bigwedge^p V^*$.

$$\omega \wedge \star\eta = \langle \omega, \eta\rangle \mathrm{d}v.$$

In particular, if $\dim V = 4$, then $\star : \bigwedge^2 V^* \to \bigwedge^2 V^*$, with $\star^2 = +1$ ($\star e^1 \wedge e^2 = e^3 \wedge e^4$, etc.). So we decompose $\bigwedge^2 V^*$ into $+$ and $-$ eigenspaces of $\star$.

$$\bigwedge^2 V^* = \bigwedge^+ V^* \oplus \bigwedge^- V^* \tag{21}$$

In fact $\dim \bigwedge^\pm V^* = 3$ with basis $\{e^1 \wedge e^2 \pm e^3 \wedge e^4,\ e^1 \wedge e^3 \pm e^4 \wedge e^2,\ e^1 \wedge e^4 \pm e^2 \wedge e^3\}$. For the induced metric on $\bigwedge^2 V^*$, (21) is an orthogonal decomposition $\big($and $|e^i \wedge e^j| = 1\big)$.

In the global version, we let (M^4, g) be an oriented Riemannian 4-manifold. Then we can decompose the bundle $\bigwedge^2 T^*M = \bigwedge^+ T^*M \oplus \bigwedge^- T^*M$, so $\Omega^2_M = \Omega^+_M \oplus \Omega^-_M$.

Definition 3.9. We say a 2-form ω is self-dual if $\star\omega = \omega$, and anti-self-dual if $\star\omega = -\omega$.

Then any $\omega = \omega^+ + \omega^-$ of SD/ASD components, and

$$|\omega|^2 = |\omega^+|^2 + |\omega^-|^2. \tag{22}$$

More generally, for a vector bundle $(E, \langle\ ,\ \rangle) \to (M^4, g)$, $\star$ extends to $\star : \Omega^2_M(E) \to \Omega^2_M(E)$ by $\star(\omega \otimes \varphi) = (\star\omega) \otimes \varphi$ and (22) holds.

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Day 16

For a connection ∇ on E, with curvature $F^\nabla \in \Omega^2(\mathfrak{g})$, $F = F^+ + F^-$, and Yang-Mills action

$$\mathcal{E}(\nabla) = \frac{1}{2}\int_M |F|^2 \mathrm{dvol}_g = \frac{1}{2}\int_M \left|F^+\right|^2 + \left|F^-\right|^2 \mathrm{dvol}_g$$

If M is compact. The chern class $c_2(E) \in H^2(M)$ (or maybe $c_2(E) - c_1^2(E)$) evaluated on the fundamental class $[M]$ is the integer

$$\begin{aligned}\kappa = c_2(E)[M] &= \frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F)\\ &= \frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge \star(F^+ - F^-)\\ &= \frac{1}{8\pi^2}\int_M \left(\langle F, F^+\rangle - \langle F, F^-\rangle\right) \mathrm{dvol}_g\\ &= \frac{1}{8\pi^2}\int_M \left(\left|F^+\right|^2 - \left|F^-\right|^2\right) \mathrm{dvol}_g\end{aligned}$$

Hence

$$\mathcal{E}(\nabla) = c_2(E)[M] + \int_M \left|F^-\right|^2 \mathrm{dvol}_g. \tag{23}$$

Note that $c_2(E)[M]$ is a topological invariant, the instanton number k. Equation (23) proves that $\mathcal{E}(\nabla) \geq c_2(E)[M] = k$ with equality iff ∇ is a self-dual connection ($F^- = 0$). In fact, we have

Lemma 3.10. *On a compact oriented Riemannian manifold M,*

$$\mathcal{E}(\nabla) \geq \left|c_2(E)[M]\right| = |k|$$

with equality iff
- ∇ *is self-dual (SD) (for* $k \geq 0$*)*
- ∇ *is anti-self-dual (ASD) (for* $k \leq 0$*)*
- ∇ *is flat (for* $k = 0$*)*

Definition 3.11. Let $\mathcal{M}^{\mathrm{SD}} := \{[\nabla] \in \mathcal{B} : F^\nabla \text{ is self-dual}\}$ be the moduli space of self-dual connections.

So $\mathcal{M}^{\mathrm{SD}} \subset \mathcal{B}$ is the absolute minimum of $E : \mathcal{B} \to \mathbb{R}$.

Remark. Locally, self-duality is a first order PDE for the connection form $F^+ = (\mathrm{d}A + A \wedge A)^+$ which implies the second order Yang-Mills equation $\mathrm{d}^* F = 0$.

There exist other interesting cases of first order equations which imply the second order field equations of a gauge theory.

3.5. Remarks on abelian gauge theories III. Moduli space.

Take the case $G = U(1)$ again, and $P \to M$ a principal $U(1)$-bundle. Every representation of $U(1)$ is a direct sum of 1-dimensional representations, each of the form $\rho_k(e^{i\theta}) = e^{ik\theta}$ for some $k \in \mathbf{Z}$. Thus

$$P \times_\rho V \cong L^{k_1} \oplus \cdots \oplus L^{k_m},$$

where $L = P \times_{\rho_1} \mathbb{C}$ is the fundamental line bundle, and $L^k = \underbrace{L \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} L}_{k}$. So without loss of generality we can consider connections on $L \to M$.

Suppose there is a solution ∇^0 of the Yang-Mills equation:

$$\mathrm{d}^* F = 0 \tag{24}$$

Any other $\nabla = \nabla^0 + iA$, with $A \in \Omega^1_M$, is Yang-Mills iff F satisfies

$$0 = (\mathrm{d}^\nabla)^* F = \mathrm{d}^* F = \underbrace{\mathrm{d}^* F^{\nabla^0}}_{=0} + i\mathrm{d}^*\mathrm{d}A$$

Let $\mathcal{S}^0 = \{\nabla^0 + iA \ : \ \mathrm{d}^* A = 0\}$ be the slice through ∇^0. If we denote $YM = \{\nabla \in \mathcal{A} \ : \ \mathrm{d}^* F^\nabla = 0\}$, then

$$\begin{aligned} YM \cap \mathcal{S}^0 &= \{\nabla^0 + iA \ : \ \mathrm{d}^*\mathrm{d}A = 0 \text{ and } \mathrm{d}\mathrm{d}^* A = 0\} \\ &= \{\nabla^0 + i\mathscr{H}^1\} \text{ where } \mathscr{H}^1 = \{\text{harmonic 1-forms}\} \cong H^1(M, \mathbb{R}). \end{aligned}$$

Thus, the moduli space

$$\mathcal{M} = \frac{YM}{\mathcal{G}} \cong \frac{YM \cap \mathcal{S}^0}{H^1(M, \mathbf{Z})} \cong \frac{H^1(M, \mathbb{R})}{H^1(M, \mathbf{Z})}$$

Proposition 3.12. *For a complex hermitian line bundle over a compact M, the moduli space $\mathcal{M}$ is either empty or (diffeomorphic to) a torus T^{b^1}, where $b^1 = \dim H^1(M, \mathbb{R})$.*

3.6. **Conformal Invariance.**

The moduli space of solutions to the Yang-Mills equation ($\mathrm{d}^* F = 0$) depends implicitly on the metric g, $\mathcal{M} = \mathcal{M}(E, g)$. Recall that two metrics g, g' are <u>conformal</u> if $g' = \phi^2 g$ for some $\phi \in C^\infty(M), \phi > 0$. (This changes "scales" but keeps angle measurements the same) Locally on M^n,

$$\int_U \left|F^\nabla\right|_g^2 \mathrm{dvol}_g = \int_U g^{ik} g^{jl} F_{ij} F_{kl} \sqrt{\det g_{ij}} \mathrm{d}x^1 \cdots \mathrm{d}x^n.$$

so

$$\begin{aligned} \int_U \left|F^\nabla\right|_{g'}^2 \mathrm{dvol}_{g'} &= \int_U \phi^{-2}\phi^{-2} g^{ik} g^{jl} F_{ij} F_{kl} \sqrt{\phi^{2n} \det g_{ij}} \mathrm{d}x^1 \cdots \mathrm{d}x^n \\ &= \int_U \phi^{n-4} |F|_g^2 \mathrm{dvol}_g. \end{aligned}$$

Lemma 3.13. *On an oriented Riemannian 4-manifold (not necessarily compact), the Yang-Mills energy, Yang-Mills equation, SD/ASD equation, and Yang-Mills moduli space are independent of the metric in its conformal class.*

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Day 17

3.7. **The standard instanton.**

We will now describe an explicit solution to the Yang-Mills equation for $G = \mathrm{SU}(2)$.

Background

The set of quaternions $\mathbb{H} = \{q = a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ has an inner-product $\langle q, r\rangle = \mathrm{Re}\,(q\overline{r})$, where $\overline{q} = a - bi - cj - dk$. The unit sphere (the unit quaternions) $S^3 \subset \mathbb{H}$ is a Lie group with Lie algebra

$$T_1S^3 = \mathrm{im}\,\mathbb{H} = \{bi + cj + dk\}.$$

We can also identify $\mathbb{H} = \mathbb{C}^2$,

$$q = a + bi + cj + dk = (a + bi) + (c + di)j = \alpha + \beta j$$

$$q \longleftrightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Then right multiplication by i gives

$$(\alpha + \beta j)i = \alpha i - (\beta i)j$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longrightarrow \begin{pmatrix}i\alpha\\ -i\beta\end{pmatrix}$$

which is given by $\gamma_i = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$. Similarly, right multiplication by $1, i, j, k$ are

$$\gamma_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \gamma_i = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \gamma_j = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \gamma_k = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

Note $\gamma_k(q) = qk = qij = \gamma_j\gamma_i(q)$. In particular, the Lie algebra $T_1S^3 = \mathrm{span}\{\gamma_i, \gamma_j, \gamma_k\} = \mathfrak{su}(2) \hookrightarrow \mathrm{End}(\mathbb{C}^2)$, the traceless skew-hermitian matrices. Hence $S^3 = \mathrm{SU}(2)$.

|

Now consider the trivial $\mathbb{H}$-line bundle

$$\begin{array}{c} E = \mathbb{R}^4 \times \mathbb{C}^2 = \mathbb{R}^4 \times \mathbb{H}\\ \big\downarrow\\ \mathbb{R}^4\end{array}$$

with the standard metric on $\mathbb{R}^4$ and $\langle (x, q), (x, r)\rangle = \mathrm{Re}\,(q\overline{r})$ on E. Every connection has the form $\nabla = \mathrm{d} + A$, with A an $\mathrm{im}\,\mathbb{H}$-valued 1-form.

The <u>standard instanton</u> is $\nabla = \mathrm{d} + A'$, where

$$A'(x) = \frac{1}{1 + |x|^2}(\alpha_1\gamma_i + \alpha_2\gamma_j + \alpha_3\gamma_k),\ x \in \mathbb{R}^4$$

where the α_i are the 1-forms

$$\alpha_1 = x^1\mathrm{d}x^2 - x^2\mathrm{d}x^1 + x^3\mathrm{d}x^4 - x^4\mathrm{d}x^3$$
$$\alpha_2 = x^1\mathrm{d}x^3 - x^3\mathrm{d}x^1 + x^4\mathrm{d}x^2 - x^2\mathrm{d}x^4$$
$$\alpha_3 = x^1\mathrm{d}x^4 - x^4\mathrm{d}x^1 + x^2\mathrm{d}x^3 - x^3\mathrm{d}x^2$$

One computes the curvature $F = \mathrm{d}A + [A \wedge A]$ (recall that our bracket has a $\frac{1}{2}$ built in) and finds

$$F = \frac{1}{\left(1+|x|^2\right)^2}\left(\mathrm{d}\alpha_1\gamma_i + \mathrm{d}\alpha_2\gamma_j + \mathrm{d}\alpha_3\gamma_k\right)$$

$$\left.\begin{aligned} \mathrm{d}\alpha_1 &= 2\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4\right) \\ \mathrm{d}\alpha_2 &= 2\left(\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^4 \wedge \mathrm{d}x^2\right) \\ \mathrm{d}\alpha_3 &= 2\left(\mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) \end{aligned}\right\} \text{SD basis on } \mathbb{R}^4$$

[Check: insert computation] So F is self-dual, and thus a Yang-Mills field (a Yang-Mills field refers to the curvature of a Yang-Mills connection).

Note that

$$|\mathrm{d}\alpha_1|^2 = 4 \cdot 2 = 8$$

$$|\gamma_i|^2 = \operatorname{tr}\left(\overline{\gamma}_i^t \gamma_i\right) = \operatorname{tr}\left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) = \operatorname{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2$$

[Check this] and the same for j, k. So the energy density is

$$\begin{aligned} e(F) &= |F|^2 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ &= \frac{48}{\left(1+|x|^2\right)^4}\, \mathrm{d}\mathbf{x} \end{aligned}$$

which is bump-like, centered at the origin.

We can build other solutions by applying conformal transformations.

(1) Dilation: The map $\rho_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ for $\lambda > 0$ by $\rho_\lambda(x) = \lambda x$ is conformal. So $A^\lambda = \rho_\lambda^* A', \nabla^\lambda = \mathrm{d} + A^\lambda$ is a self dual connection for every $\lambda > 0$.

$$\begin{aligned} A^\lambda &= \frac{1}{\lambda^2 + |x|^2}\left(\alpha_1\gamma_i + \alpha_2\gamma_j + \alpha_3\gamma_k\right) \\ F^\lambda &= \frac{\lambda^2}{\left(\lambda^2+|x|^2\right)^2}\left(\mathrm{d}\alpha_1\gamma_i + \mathrm{d}\alpha_2\gamma_j + \mathrm{d}\alpha_3\gamma_k\right) \\ e(F^\lambda) &= \left|F^\lambda\right|^2 \mathrm{d}x = \frac{48\lambda^4}{\left(\lambda^2+|x|^2\right)^2}\, \mathrm{d}x \end{aligned}$$

(Check this)

Lemma 3.14. *As $\lambda \to 0$, $e(F^\lambda) \to 8\pi^2 \cdot \delta(0)$ as distributions.*

Proof. Let $u = \lambda^2 + r^2$ $(r = |x|, \mathrm{d}u = 2r\mathrm{d}r)$. Note $\mathrm{Vol}(S^3) = 2\pi^2$.

$$\int_{\mathbb{R}^4} e(F) = \mathrm{Vol}(S^3)\int_0^\infty 48\frac{\lambda^4}{u^4}\underbrace{r^3\mathrm{d}r}_{\frac{1}{2}(u-\lambda^2)\mathrm{d}u} = 48\pi^2\lambda^4\int_{\lambda^2}^\infty u^{-3} - \lambda^2 u^{-4}\, \mathrm{d}u$$

$$= 48\pi^2\lambda^4\left[\frac{-1}{2u^2} + \frac{\lambda^2}{3u^3}\right]_{\lambda^2}^\infty = 48\pi^2\lambda^4\left[\frac{1}{2\lambda^4} - \frac{\lambda^2}{3\lambda^6}\right] = 48\pi^2\left[\frac{1}{2} - \frac{1}{3}\right] = 8\pi^2.$$

Now, for every $\varepsilon > 0$,

$$\int_{B^c(\varepsilon)} e(F^\lambda) \xrightarrow{\lambda\to 0} 0 \text{ and } \int_{\mathbb{R}^4} xe(F^\lambda) \longrightarrow 0.$$

(Check this) For every $f \in C_c^\infty(\mathbb{R}^4)$, write $f(x) = f(0) + xg(x)$ for a bounded function g.

$$\int_{\mathbb{R}^4} f(x)e(F^\lambda) = f(0)\int_{\mathbb{R}^4} e(F^\lambda) + \int_{\mathbb{R}^4} xg(x)e(F^\lambda) \longrightarrow 8\pi^2 f(0).$$

□

(2) Translation: For each $y \in \mathbb{R}^4$,

$$A^y(x) = \frac{1}{\lambda^2 + |x-y|^2}(\cdots)$$

is a self dual connection.

(3) Stereographic projection: $\sigma^\pm : S^4 - \{\pm\} \to \mathbb{R}^4$ is a conformal map with $\mathrm{d}\sigma^+_{(-p)} : T_{(-p)}S^4 \to \mathbb{R}^4$ an isometry. Hence the pullback connections

$$\begin{cases} \nabla^+ = \mathrm{d} + (\sigma^+)^* A^\lambda & \text{for } S^4 - \{+p\} \\ \nabla^- = \mathrm{d} + (\sigma^-)^* A^\lambda & \text{for } S^4 - \{-p\} \end{cases} \tag{25}$$

are self dual connections on the trivial $\mathbb{H} = \mathbb{C}^2$ vector bundles.

If we regard S^4 as the unit sphere in $\mathbb{H}\times\mathbb{R}$, define

$$g : S^4 \setminus \{\pm p\} \to \mathrm{SU}(2)$$
$$(q, z) \mapsto \gamma_q.$$

On the equator, this is the identity map $S^3 \to \mathrm{SU}(2) = S^3$. One can see that $\nabla^- = g.\nabla^+ = g\nabla^+ g^{-1}$. (Check this.) So using g as a transition map, we obtain a (twisted) $\mathbb{C}^2$-vector bundle $E \to S^4$ on which (25) defines a self-dual connection ∇^λ for each λ. Further, we can rotate S^4. This gives us a 5-parameter family of self-dual connections $\nabla^{\lambda,y}$ with scale $\lambda \in \mathbb{R}^4$ centered at $y \in S^4$.

Lemma 3.15.
(a) *For each* y, $\lim_{\lambda\to 0} e(F^{\lambda,y}) = 8\pi^2\delta(y)$.
(b) *Each* $A^{\lambda,y}$ *is a connection on the* $\kappa = 1$ *bundle on* S^4.

Proof.
(a) By rotating S^4, we can assume $y = p$. But $\mathrm{d}\sigma^+ : T_pS^4 \to T_0\mathbb{R}^4$ is an isometry (Check this). Then the result follows from lemma 3.14.
(b) By self-duality,

$$\kappa = \frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F\wedge F) = \frac{1}{8\pi^2}\int_{S^4} \left|F^{\lambda,y}\right|^2 \mathrm{dvol}_g \xrightarrow{\lambda\to 0} \frac{1}{8\pi^2}\cdot 8\pi^2 = 1$$

But this number is a topological invariant of the bundle and thus independent of λ.

□

Curvature of the standard instanton.

...

□

4. Overview of Analysis

2019/02/22, Day 18

4.1. Sobolev Spaces & Embedding Theorem.

Let $E \to (M, g)$ be a hermitian vector bundle over a compact Riemannian manifold (Lorentzian is not good enough here).

Definition 4.1. For an integer $k \geq 0$ and $1 < p \in \mathbb{R}$, the Sobolev space $W^{k,p}(E)$ is the completion of the space of smooth sections $\Gamma(E)$ under the (Sobolev) norm:

$$\|\xi\|_{k,p} = \left(\int_M \left|\nabla^k \xi\right|^p + \cdots + |\nabla \xi|^p + |\xi|^p \ \mathrm{dvol}_g \right)^{\frac{1}{p}} \tag{26}$$

where $\nabla^k \xi$ is the k^{th} covariant derivative, using a connection ∇ on E.

Remark.
- This is a Banach space (Hilbert space for $p = 2$).
- $W^{0,p}(E) = L^p(E)$.
- Let $\nabla' = \nabla + A$,with A bounded (M is compact), be another connection on E. Then there exist constants $C_1, C_2 > 0$ such that
$$C_1 \|\xi\|_{k,p,\nabla} \leq \|\xi\|_{k,p,\nabla'} \leq C_2 \|\xi\|_{k,p,\nabla}$$
i.e. the norms are equivalent, so $W^{k,p}(E)$ is defined independent of the connection ∇.

Definition 4.2.
$$C^l = \{C^l \text{ sections of } E\} = \text{completion of } \Gamma(E) \text{ under } \|\cdot\|_{C^l}$$
$$\|\xi\|_{C^l} = \sup\left(\left|\nabla^l \xi\right| + \cdots + |\nabla \xi| + |\xi|\right).$$

Recall:

- A Banach space is a vector space with a norm satisfying
$$\|\xi + \eta\| \leq \|\xi\| + \|\eta\|, \quad \|\xi\| \geq 0, \quad \|\xi\| = 0 \iff \xi = 0.$$
which is complete as a metric space.
- A linear map $\varphi : V \to W$ between Banach spaces is bounded ($\|\varphi(v)\| \leq C\|v\|$ for all v) if and only if it is continuous.
- A linear map $\varphi : V \to W$ is called compact if, for every bounded sequence $\{v_k\}$ in V, the image $\{\varphi(v_k)\}$ has a convergent subsequence in W.

Theorem 4.3 (Sobolev embedding)**.** *With $E \to M^n$ as above, the identity $\Gamma(E) \to \Gamma(E)$ induces*

(a) *a continuous linear map* $W^{k,p}(E) \to W^{l,q}(E)$ *if* $k - \frac{n}{p} \geq l - \frac{n}{q}$,
(b) *a compact linear map* $W^{k,p}(E) \to W^{l,q}(E)$ *if* $k > l$ *and* $k - \frac{n}{p} > l - \frac{n}{q}$,
(c) *a compact linear map* $W^{k,p}(E) \to C^l(E)$ *if* $k - \frac{n}{p} > l$.

The tensor product $(\xi, \eta) \to \xi \otimes \eta$ *induces*

(d) *a continuous linear map* $W^{k,p}(E) \times W^{k,p}(E) \to W^{k,p}(E \otimes E)$ *if* $k - \frac{n}{p} > 0$.

Proof. (Cited Evans - Ch. 5) For a compact manifold M, the Sobolev theorems carry over from $\mathbb{R}^n$ by using finite covers and partitions of unity.

Meanwhile here is a nice proof from [Rosenberg - Laplacian on Riemannian manifolds] using the Fourier transform, for a special case (recorded here for my own reference). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We want to prove that if $f \in W^{k,2}(\Omega)$ then $f \in C^s(\Omega)$ for all $s < k - \frac{n}{2}$. Let $\widehat{f}$ denote the Fourier transform.

The Fourier transform is an isometry on L^2, and differentiation transforms into multiplication, so the Sobolev norm is equivalent to

$$\|f\|_{k,2} \approx \left(\int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 (1 + |\xi|^2)^k \mathrm{d}\xi \right)^{\frac{1}{2}} \tag{27}$$

because one can find constants to compare the two degree k polynomials in $|\xi|^2$, namely $(1 + |\xi|^2)^k$ and $(1 + |\xi|^2 + \cdots + |\xi|^{2k})$ which appears in the original norm.

First consider the $s = 0$ case: assume $k > n/2$ and $f \in W^{k,2}(\Omega)$. Then

$$\begin{aligned} |f(x)| &= \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi) \ \mathrm{d}\xi \right| \\ &= \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1 + |\xi|^2)^{-\frac{k}{2}} (1 + |\xi|^2)^{\frac{k}{2}} \widehat{f}(\xi) \ \mathrm{d}\xi \right| \\ &\leq \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-k} \ \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 (1 + |\xi|^2)^k \ \mathrm{d}\xi \right)^{\frac{1}{2}}, \end{aligned}$$

and since $k > n/2$ the left integral is finite, so

$$|f(x)| \leq C\|f\|_{k,2}.$$

Now by definition, a function f in $W^{k,2}$ is a limit of smooth functions $f_i \to f$. This last statement shows that

$$\sup_{x\in\Omega} |f_i(x) - f(x)| \leq C\|f_i - f\|_{k,2} \longrightarrow 0$$

i.e. that $f_i \to f$ uniformly, and thus f is continuous.

Now we look at general s. For any multi-index α, the basic derivative operator $\partial^\alpha : W^{k,2} \to W^{k-|\alpha|,2}$ is bounded (see below). Assume $f \in W^{k,2}$ and $k - s > \frac{n}{2}$. Then for any α with $|\alpha| \leq s$ (i.e. take fewer than s derivatives), the first part shows that $\partial^\alpha f \in C^0$. Thus $f \in C^s$.

To see explicitly why ∂^α is bounded,

$$\begin{aligned} \|\partial^\alpha f\|^2_{k-|\alpha|} &= \int_{\mathbb{R}^n} \left|\widehat{\partial^\alpha f}\right|^2 (1 + |\xi|^2)^{k-|\alpha|} \ \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\widehat{f}|^2 (1 + |\xi|^2)^{k-|\alpha|} \ \mathrm{d}\xi \\ &\leq C\|f\|_{k,2} \end{aligned}$$

since we can find C such that
$$|\xi^\alpha|(1+|\xi|^2)^{k-|\alpha|} \leq C(1+|\xi|^2)^k.$$
(Remember to think about $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$ and $|\xi|^2 = \xi_1^2+\cdots+\xi_n^2$.)

(Check: Also insert proof of Rellich-Kondrachov compactness: $W^{k,2} \hookrightarrow W^{l,2}$ is compact if $k > l$.)

Note that (27) can be used to define Sobolev space for any real k, and the proofs above work for all k. □

4.2. **Elliptic Linear Differential Operators.**

The prime example of an elliptic operator is the laplacian Δ. Consider, in general, a linear differential operator $D : \Gamma(E) \to \Gamma(F)$, between two vector bundles over M.

Definition 4.4. Ellipticity is a requirement on the coefficients of the top order term: In local coordinates
$$\begin{aligned} \text{If} \quad & D = \sum a^i \nabla_{\partial_i} + A \\ \text{or} \quad & D = \sum a^{ij}\nabla_i\nabla_j + \sum A^i \nabla_{\partial_i} + B \\ \text{or} \quad & D = \sum a^I \nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_l} + \text{lower order terms} \end{aligned}$$
Then D is called elliptic if $a^I(x) : E_x \to F_x$ is an isomorphism for all $x \in M$ and all I.

A good reference is [Nicolaescu - geo of manifolds, chapter 10]

A few examples of elliptic operators:

Example 4.5. A Dirac-type (1st order) operator $D = \not{D} + A$.

Example 4.6. A Laplace-type (2nd order) operator $D = \nabla^*\nabla + A\cdot\nabla + B$.

Example 4.7. In an orthonormal frame, the rough laplacian $\nabla^*\nabla$ has (a^{ij}) equal to the identity matrix.

We will use, without proof, the following important result:

Theorem 4.8 (Elliptic regularity I)**.** *Let D be an l^{th} order elliptic operator with C^∞ coefficients. If $\xi \in W^{k,p}$ and $D\xi \in W^{k,p}$ then $\xi \in W^{k+l,p}$ and*
$$\|\xi\|_{k+l,p} \leq C\left(\|\xi\|_p + \|D\xi\|_{k,p}\right). \tag{28}$$

Note (to self).
- $\|\xi\|_p := \|\xi\|_{0,p} = \|\xi\|_{L^p}$ is the L^p norm.
- To be careful, I think we need that ξ is in the domain of D (in the sense of an unbounded operator) in order to write down $D\xi$ at all. Response: We assumed that $D\xi \in W^{k,p}$ to start. In particular, ξ is assumed to be in the domain of D.

- With an induction proof (on k), the above estimate is equivalent to

$$\|\xi\|_{k+l,p} \leq C\left(\|\xi\|_{k,p} + \|D\xi\|_{k,p}\right) \tag{29}$$

which also appears as "elliptic regularity" (e.g. in Rosenberg - The Laplacian on a Riemannian Manifold).

Proof. The better bound is (28), so the harder direction is going back to that one. If $k = 0$, then the inequalities are the same. Suppose for some k, we have proven that (28) is true for all $j \leq k$. We also assume (29–with k replaced by h), for all integers $h \geq 0$. In particular we assume

$$\|\xi\|_{k+1+l,p} \leq C\left(\|\xi\|_{k+1,p} + \|D\xi\|_{k+1,p}\right).$$

But by the induction hypothesis with $j = k+1-l \leq k$ (assuming reasonably that $l \geq 1$) we can bound the one different term

$$\|\xi\|_{k+1=j+l,p} \leq C\left(\|\xi\|_{p} + \|D\xi\|_{j,p}\right).$$

Then since $j \leq k$ again, $\|\cdot\|_{j,p} \leq \|\cdot\|_{k,p}$, so we have the desired

$$\|\xi\|_{k+1+l,p} \leq C\left(\|\xi\|_{p} + \|D\xi\|_{k+1,p}\right).$$

$\square$

- As noted in Rosenberg, (at least over a compact manifold) the reverse inequality is true with a different constant C, so the two sides of the inequality define equivalent norms.

(Check: insert proof of Gårding's inequality from Rosenberg, uses Weitzenböck formula. This is for the special case of $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ on differential forms.)

Applications

Corollary 4.9. *With D as above, every $W^{1,2}$ solution of $D\xi = 0$ or $D\xi = \lambda\xi$ is C^∞.*

Proof. (Bootstrapping) Repeatedly apply (28) with D or $D - \lambda I$:

$$\begin{aligned}
\xi \in W^{1,2} &\implies \xi \in W^{1+l,2} \\
&\implies \xi \in W^{1+2l,2} \\
&\quad\vdots \\
&\implies \xi \in W^{m,2}, \text{ for every } m, \text{ (note that } W^{t,p} \hookrightarrow W^{s,p} \text{ easily whenever } t > s,). \\
&\implies \xi \in C^{m-\frac{n}{2}}, \text{ for every } m \text{ by Sobolev embedding.} \\
&\implies \xi \in C^\infty
\end{aligned}$$

$\square$

(Check: Why do we need $D - \lambda I$? Why not just replace $\|D\xi\|_{1,2} = \lambda\|\xi\|_{1,2} < \infty$?)

Corollary 4.10. *Each eigenspace $E_\lambda = \{\xi \in W^{1,2} : D\xi = \lambda\xi\}$ and $E_\lambda^- = \mathrm{span}\{E_\mu : \mu \leq \lambda\}$ is finite dimensional.*

Proof. Fix λ and consider the unit sphere $S_\lambda = \{\xi \in E_\lambda : \|\xi\|_{1,2} = 1\}$. Let ξ_n be a sequence in S_λ. By (28),

$$\|\xi_n\|_{1+k,2} \leq C\left(\|\xi_n\|_2 + \|\lambda\xi_n\|_{1,2}\right) \leq C(1+\lambda)\|\xi_n\|_{1,2} < C',$$

the sequence is bounded in $W^{1+k,2}$. But $W^{1+k,2} \hookrightarrow W^{1,2}$ is a compact embedding, so there is a subsequence (still called ξ_n) converging to ξ_0 in $W^{1,2}$. The norm is continuous so $\|\xi_0\|_{1,2} = 1$, i.e. $\xi_0 \in S_\lambda$.
This proves that the unit sphere S_λ is compact, so E_λ is finite dimensional. □

(Check: Significance & proof of the second E_λ^- result?)

Corollary 4.11 (Poincaré Inequality)**.** *With D as above, and k, p such that $W^{k,p} \hookrightarrow L^2$, there is a C such that*

$$\|\xi\|_{k+l,p} \leq C\|D\xi\|_{k,p} \quad \text{for } \xi \perp_{L^2} \ker D \tag{30}$$

for $\xi \in W^{k+l,p}$ which are L^2-perpendicular to $\ker D$. Thus only the $D\xi$ term in (28) is needed for such ξ.

Proof. Since p is fixed we will leave this off the norms.
Suppose this was not true. That is, suppose, no matter how big C is, there is some ξ which breaks (30). In particular, for each $C = n \in \mathbf{N}$, there is a $\xi_n \in W^{k+l,p} \cap (\ker D)^{\perp_{L^2}}$ such that

$$\|\xi_n\|_{k+l} > n\|D\xi_n\|_k$$

Note that if we replace $\xi_n \mapsto \xi_n/\|\xi_n\|_{k+l}$, then the above is unchanged. So we have a sequence ξ_n such that

$$\|\xi_n\|_{k+l} = 1$$
$$\|D\xi_n\|_k < \frac{1}{n}.$$

(Again) the compact embedding $W^{k+l,p} \hookrightarrow W^{k,p}$ gives us that a subsequence ξ_n converges in $W^{k,p}$ to some ξ_0. Then

$$\begin{aligned}\|\xi_n - \xi_m\|_{k+l} &\leq C\left(\|\xi_n - \xi_m\|_k + \|D\xi_n - D\xi_m\|_k\right)\\ &\leq C\left(\|\xi_n - \xi_m\|_k + \|D\xi_n\|_k + \|D\xi_m\|_k\right) \to 0\end{aligned}$$

shows that the subsequence we picked is in fact Cauchy, and thus convergent to ξ_0' in $W^{k+l,p}$. The two limits must be the same, since

$$\begin{aligned}\|\xi_0 - \xi_0'\|_{k,p} &\leq \|\xi_0 - \xi_n\|_k + \|\xi_0' - \xi_n\|_k\\ &\leq \|\xi_0 - \xi_n\|_k + \|\xi_0' - \xi_n\|_{k+l}\end{aligned}$$

can be made arbitrarily small.

Now

$$\begin{aligned}\|D\xi_0\|_k &\leq \|D\xi_n\|_k + \|D(\xi_0 - \xi_n)\|_k\\ &\leq \frac{1}{n} + C\|\xi_0 - \xi_n\|_{k+l} \quad \longrightarrow 0\end{aligned}$$

(we used that $D : W^{k+l,p} \to W^{k,p}$ is bounded, a believable statement which we prove below). So $\xi_0 \in \ker D$. Our initial assumption implies

$$0 = \langle \xi_n, \xi_0 \rangle_{L^2} \longrightarrow \|\xi_0\|_{L^2}^2.$$

Therefore $\xi_0 = 0$ and $\xi_n \to 0$ in $W^{k+l,p}$ contradicting that $\|\xi_n\|_{k+l} = 1$. □

As mentioned, D is bounded:

Lemma 4.12. *For (M,g) compact and $D : \Gamma(E) \to \Gamma(F)$ an l^{th}-order linear differential operator, with C^∞ coefficients. Then D extends to a bounded linear map*

$$D : W^{k+l,p}(E) \to W^{k,p}(F) \tag{31}$$

Proof. The idea is quick for smooth sections $\Gamma(E)$: bound the coefficients by compactness. After that we approximate for all of $W^{k+l,p}$.
Fix a connection ∇ on E. Write locally

$$\begin{aligned} D &= a^{i_1,\cdots,i_l}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_l} + \cdots \\ &= \sum_{j=1}^{l}\sum_{|I|=j} A^I \nabla_I^j. \end{aligned}$$

The A^I are smooth bundle maps. Because M is compact, $\left\|A^I\right\| \le C$ for some constant. For smooth ξ,

$$\begin{aligned} \|D\xi\|_k &\le \sum_{j,I}\left\|A^I \nabla_I^j \xi\right\|_k \\ &\le C\sum_{j,I}\left\|\nabla_I^j \xi\right\|_k \\ &\le \|\xi\|_{k+l} \end{aligned}$$

We extend D to the Sobolev space (which is the completion): Any $\xi \in W^{k+l,p}$ is the $W^{k+l,p}$ limit of smooth ξ_n. Then

$$\|D(\xi_n - \xi_m)\|_k \le C\|\xi_n - \xi_m\|_{k+l} \longrightarrow 0$$

so $D\xi_n$ is Cauchy, with limit denoted by $D\xi$ (this *defines* $D\xi$ on $W^{k+l,p}$). This is independent of the choice of sequence and satisfies the same boundedness. (Check this) □

Exercise 4.13 (Interpolation Inequality)**.** For each ε, $1 \le l < k$, there is a constant $C(\varepsilon)$ such that

$$\|\xi\|_{l,p} \le \varepsilon \underbrace{\|\xi\|_{k,p}}_{\text{stronger}} + C(\varepsilon)\underbrace{\|\xi\|_{L^1}}_{\text{weaker}} \tag{32}$$

[Check - move this to earlier, right after Sobolev spaces?]

Proof 1. The intermediate terms in the definition of the Sobolev norm can be bounded by the top and bottom terms

$$\|\xi\|_{k,p} \approx \left(\int_M \left|\nabla^k \xi\right|^p\right)^{\frac{1}{p}} + \|\xi\|_{L^1}$$

[Check - detail proof] □

Proof 2 (Exercise). Suppose not. Then there is an ε and a sequence $\xi_n \in W^{k,p}$ such that

$$\|\xi_n\|_{l,p} > \varepsilon\|\xi_n\|_{k,p} + n\|\xi_n\|_{L^1}$$

for each n. By homogeneity of the norms, we can divide and assume $\|\xi_n\|_{l,p} = 1$. Thus

$$1 > \varepsilon\|\xi_n\|_{k,p} + n\|\xi_n\|_{L^1}.$$

Now

$$1 > \varepsilon\|\xi_n\|_{k,p}$$

implies a subsequence converges to some ξ in $W^{l,p}$. Notice that $W^{l,p} \hookrightarrow L^1$ continuously, so it also converges in L^1. Meanwhile

$$1 > n\|\xi_n\|_{L^1}$$

gives us that $\|\xi_n\|_{L^1} \to 0$, which implies our limit $\xi = 0$, contradicting that

$$\|\xi\|_{l,p} = \lim_{n\to\infty} \|\xi_n\|_{l,p} = 1.$$

□

4.3. **Weak Solutions.**

The theme of "Elliptic Regularity" requires the following

Definition 4.14. The (L^2) adjoint D^* of D is defined by

$$\langle D^*\xi, \eta\rangle_{L^2} = \langle \xi, D\eta\rangle_{L^2}$$

$$\int_M \langle D^*\xi, \eta\rangle \text{ dvol} = \int_M \langle \xi, D\eta\rangle \text{ dvol}$$

for all smooth ξ, η.

Example 4.15. (Exterior derivative) The adjoint of $\mathrm{d} : \Omega^p_M \to \Omega^{p+1}_M$ is defined by the above and is an operator $\mathrm{d}^* : \Omega^{p+1}_M \to \Omega^p_M$.
The hodge star tells us what this operator is explicitly. Let $\xi \in \Omega^p$ and $\eta \in \Omega^{p+1}$. Recall that $\star^2 = (-1)^{p(n-p)}$ on p-forms (as well as $(n-p)$-forms). Then

$$\begin{aligned}\int_M \langle \mathrm{d}\xi, \eta\rangle \text{ dvol} &= \int_M \mathrm{d}\xi \wedge \star\eta \\ &= (-1)^{p-1}\int_M \xi \wedge \mathrm{d}\star\eta \\ &= (-1)^{p-1+p(n-p)}\int_M \xi \wedge \star\star\,\mathrm{d}\star\eta \\ &= (-1)^{pn-1}(-1)^{p-p^2}\int_M \langle \xi, \star\,\mathrm{d}\star\eta\rangle \text{ dvol} \\ &= (-1)^{pn-1}\int_M \langle \xi, \star\,\mathrm{d}\star\eta\rangle \text{ dvol}\end{aligned}$$

shows the formula

$$\mathrm{d}^* = (-1)^{pn-1} \star \mathrm{d} \star \tag{33}$$

when acting on $(p+1)$-forms.

In general, if D is a k^{th} order linear differential operator (LDO) with smooth coefficients, then by integration by parts so is D^*.

Definition 4.16. A section $\xi \in W^{k,p}(E)$ is a weak solution of $D\xi = \eta$ if

$$\int_M \langle \xi, D^* \varphi \rangle - \langle \eta, \varphi \rangle = 0$$

for all smooth φ. (Imagine $\langle D\xi - \eta, \varphi \rangle = 0$.)

Now elliptic regularity applies starting with weak solutions for elliptic (LDO) with smooth coefficients.

Example 4.17. If $f \in W^{1,2}$ is a weak solution of $Df = 0$, then f is smooth.

4.4. Hölder Spaces.

For a Riemannian manifold M, recall that

$$C^0(M) = \{\text{bounded continuous functions } f : M \to \mathbb{R}\}$$

is a Banach space under the norm

$$\|f\|_{C^0} = \sup_{x \in M} |f(x)|$$

Hölder spaces extend this notion, and interpolate between C^0, Lipschitz, and C^1 functions.

Definition 4.18. Let $\alpha \in [0,1]$. A function $f : M \to \mathbb{R}$ is α-Hölder continuous if there exists a C such that

$$|f(x) - f(y)| \leq C d(x,y)^\alpha$$

for all $x, y \in M$, (notice for $\alpha = 1$ this is just Lipschitz). Then the α-Hölder space

$$C^{0,\alpha}(M) = \{f \in C^0 \ : \ f \text{ is } \alpha\text{-Hölder continuous}\}$$

is a Banach space with the norm

$$\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

Similarly, using

$$\|f\|_{C^k} = \sup_{\substack{|\alpha| = k \\ x \in M}} |\partial^\alpha f(x)|$$

we can define the space $C^{k,\alpha}$ using the norm

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \sum_{|\alpha| = k} \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{d(x,y)^\alpha}$$

All of this works for sections of vector bundles, where we define $|\partial^\alpha f(x) - \partial^\alpha f(y)|$ using parallel transport.

Theorem 4.19 (Elliptic Regularity II)**.** *Let D be an l^{th} order LDO with smooth coeffictients. Then $\xi \in C^\alpha$ and $D\xi \in C^{k,\alpha}$ implies that $\xi \in C^{k+l,\alpha}$ and*

$$\|\xi\|_{C^{k+l,\alpha}} \leq C \left(\|D\xi\|_{C^{k,\alpha}} + \|\xi\|_{C^\alpha} \right). \tag{34}$$

One obtains elliptic regularity for non-linear equations by bootstrapping again.

Example 4.20. Suppose $\varphi \in L^6$ satisfies the equation

$$D\varphi = \varphi^2$$

weakly on a compact manifold M^4, with D a first order elliptic operator. Then

$$\begin{aligned}
\varphi^2 \in L^3 &\implies D\varphi \in L^3 \\
\text{Elliptic Regularity I, (28)} &\implies \varphi \in W^{1,3} \\
\text{Sobolev embedding: } \left(k - \frac{n}{p} \geq l - \frac{n}{q}\right) &\rightsquigarrow \left(1 - \frac{4}{3} \geq 0 - \frac{4}{q}\right) \\
&\implies \varphi \in L^{12} \\
\text{Repeat: } \varphi^2 \in L^6 &\implies D\varphi \in L^6 \\
&\implies \varphi \in W^{1,6} \\
\left(1 - \frac{4}{6} = \frac{1}{3} > 0\right) &\implies \varphi \in C^{0,\alpha}, \quad \alpha = \frac{1}{3} \\
\text{Repeat: } \varphi^2 \in C^{0,\alpha} &\implies D\varphi \in C^{0,\alpha} \\
\text{Elliptic Regularity II, (34)} &\implies \varphi \in C^{1,\alpha} \\
&\quad \vdots \\
&\implies \varphi \in C^\infty
\end{aligned}$$

4.5. **Variational Methods.**

The goal here is to find weak solutions to a differential equation by minimizing an energy functional.

Let $D : \Gamma(E) \to \Gamma(F)$ be a first order elliptic operator on a compact manifold M. Then D^*D is a second order, self-adjoint operator, so we can consider its eigenspaces. Define the energy

$$E(\xi) = \int_M |D\xi|^2.$$

The energy is a bounded quadratic, thus smooth, function on $W^{1,2}$. The statement of elliptic regularity becomes

$$\begin{aligned}
\|\xi\|_{1,2}^2 &\leq C\left(\|D\xi\|_{L^2}^2 + \|\xi\|_{L^2}^2\right) \\
&= C\left(E(\xi) + \|\xi\|_{L^2}^2\right)
\end{aligned}$$

Lemma 4.21 (Minimization Lemma). *Let $V \subset L^2$ be a closed linear subspace, and let $S \subset L^2$ be the unit sphere. If $V \cap S \neq \varnothing$, then E attains its minimum on $S_V = V \cap S \cap W^{1,2}$*

Proof. Choose a minimizing sequence $\xi_n \in S_V$

$$E(\xi_n) \to E_0 = \inf\{E(\xi) \ : \ \xi \in S_V\}.$$

By elliptic regularity ξ_n is bounded in $W^{1,2}$ and thus has a weakly convergent subsequence, converging strongly in L^2 by compactness $W^{1,2} \hookrightarrow L^2$. So the $W^{1,2}$ limit ξ_0 is in V by closure and in S by strong L^2

convergence (which implies convergence of the norms). Therefore ξ_0 in S_V. Now

$$\begin{aligned} E(\xi_0) - E(\xi_n) &= \int |D\xi_0|^2 - |D\xi_n|^2 \\ &= \int 2|D\xi_0|^2 - |D\xi_0|^2 - 2\langle D\xi_0, D\xi_n\rangle + 2\langle D\xi_0, D\xi_n\rangle - |D\xi_n|^2 \\ &= \int 2\langle D\xi_0, D(\xi_0 - \xi_n)\rangle - |D(\xi_0 - \xi_n)|^2 \\ &\leq \int 2\langle D\xi_0, D(\xi_0 - \xi_n)\rangle \longrightarrow 0 \end{aligned}$$

implies that

$$E_0 \leq E(\xi_0) \leq \liminf E(\xi_n) = E_0.$$

Thus the minimum is attained by $\xi_0 \in S_V$. □

Theorem 4.22 (Spectral Theorem for D^*D)**.** *With D an elliptic first order differential operator as above, there is a complete L^2-orthogonal decomposition*

$$L^2(E) = \bigoplus E_{\lambda_i}$$

into finite dimensional eigenspaces ($D^*D\xi = \lambda\xi$). The eigenfunctions ξ are smooth, and the eigenvalues $\{\lambda_i\}$ are real, nonnegative, with no accumulation points.

Proof. Apply the minimization lemma repeatedly to obtain the eigenspaces:

Set $V_1 = L^2$ to get a minimizer $\xi_1 \in S_1 := S_{V_1}$.
Set $V_2 = \{\xi \in L^2 \ : \ \langle \xi, \xi_1\rangle_{L^2} = 0\}$, the L^2-orthogonal space to ξ_1, with minimizer $\xi_2 \in S_2$.
In general, set $V_n = [\text{span}(\xi_1, \dots, \xi_{n-1})]^\perp$, with minimizer ξ_n. So we have a sequence $\xi_n \in S_n$ with increasing energy

$$E(\xi_1) \leq E(\xi_2) \leq \dots$$

Orthogonality of the $\{\xi_n\}$ is clear from the construction. Let $W = \overline{\text{span}\{\xi_n\}}^\perp$... [Check: Finish proof of $\overline{\text{span}\{\xi_n\}} = L^2$]

The numbers $\lambda_n := E(\xi_n) = \int |D\xi_n|^2$ are real and nonnegative. Since the ξ_n's were chosen as minimizers, we can look at the variation of the energy functional at ξ_n:

$$0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E(\xi_n + t\psi)\Big|_{t=0} = \int \langle D\xi_n, D\psi\rangle$$
$$\left(= \int \langle D^*D\xi_n, \psi\rangle \right)$$

for every $\psi \in TS_{V_n}$. By L^2-orthogonality we have

$$\int \langle D\xi_j, D\xi_k\rangle = 0 \quad \text{for all } j \neq k.$$

Now take any $W^{1,2}$ section ψ, and by completeness write $\psi = \sum a_n \xi_n$. We have

$$\begin{aligned}\int \langle D\xi_n, D\psi\rangle - \langle \lambda_n \xi_n, \psi\rangle &= \int \langle D\xi_n, a_n D\xi_n\rangle - \lambda_n \langle \xi_n, a_n \xi_n\rangle \\ &= a_n(E(\xi_n) - \lambda_n) = 0,\end{aligned}$$

i.e. $D^*D\xi_n = \lambda_n \xi_n$ weakly. By elliptic regularity the eigenfunctions are smooth. [Check: write out details]

[Check: $\dim E_{\lambda_i} < \infty$ and no accumulation points.] □

Say the operator above is defined as $D : \Gamma(E) \to \Gamma(F)$. The operator extends to $W^{1,2}(E) \to L^2(F)$. So we have

$$\bigoplus E_{\lambda_i} \underset{D^*}{\overset{D}{\rightleftarrows}} \bigoplus F_{\mu_i}$$

Notice if $D^*D\xi = \lambda\xi$, then

$$\begin{aligned} D(D^*D\xi) &= D(\lambda\xi) \\ (DD^*)(D\xi) &= \lambda(D\xi)\end{aligned}$$

so $\zeta = D\xi$ is an eigenvector for DD^* with the same eigenvalue λ.

Furthermore, if $\lambda \neq 0$, then

$$\int |D\xi|^2 = \int \langle D^*D\xi, \xi\rangle = \int \lambda |\xi|^2 \tag{35}$$

and thus $\zeta = D\xi \neq 0$. This means that D maps $E_\lambda \to F_\lambda$ isomorphically.

Each E_λ for $\lambda \neq 0$, is "paired" by D with an F_λ. The full picture is

$$\ker(D^*D) \oplus \bigoplus_{\lambda_i \neq 0} E_{\lambda_i} \xrightarrow{D} \bigoplus_{\lambda_i \neq 0} F_{\lambda_i} \oplus \ker(DD^*).$$

In fact the trick in (35) shows that $\ker(D^*D) = \ker(D)$ and

$$\ker(DD^*) = \ker(D^*) = (\operatorname{im} D)^\perp = \operatorname{coker}(D),$$

so

$$\ker(D) \oplus \bigoplus_{\lambda_i \neq 0} E_{\lambda_i} \xrightarrow{D} \bigoplus_{\lambda_i \neq 0} F_{\lambda_i} \oplus \operatorname{coker}(D).$$

Remark. We can modify the decomposition to have eigenvalues of D rather than D^*D. Define the first order operator

$$\widehat{D} : \Gamma(E \oplus F) \to \Gamma(E \oplus F)$$

$$\widehat{D}\begin{pmatrix}\xi \\ \psi\end{pmatrix} = \begin{pmatrix}0 & D^* \\ D & 0\end{pmatrix}\begin{pmatrix}\xi \\ \psi\end{pmatrix} = \begin{pmatrix}D^*\psi \\ D\xi\end{pmatrix}.$$

Then for $\xi \in E_\lambda$,

$$\widehat{D}\begin{pmatrix}\xi \\ \frac{\pm 1}{\sqrt{\lambda}} D\xi\end{pmatrix} = \begin{pmatrix}\frac{\pm 1}{\sqrt{\lambda}} D^*D\xi \\ D\xi\end{pmatrix} = \begin{pmatrix}\pm\sqrt{\lambda}\xi \\ D\xi\end{pmatrix} = \pm\sqrt{\lambda}\begin{pmatrix}\lambda\xi \\ \frac{\pm 1}{\sqrt{\lambda}} D\xi\end{pmatrix}.$$

So $\widehat{D}$ is self-adjoint with eigenvalues equal to the square roots of the eigenvalues of D^*D.

4.6. **Spectral Flow and Index.**

Now we will consider a family of elliptic operators D_p, in general parameterized by a manifold $p \in P$.

Example 4.23. Define D using a family of coefficient matrices
$$\{A_p : \mathbb{R}^n \to \mathbb{R}^m \ : \ p \in P = \mathbb{R}^l\},$$
so that
$$D_p = (A_p)^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}.$$

Example 4.24. For a family of connections $P \subset \mathcal{A}$ on a vector bundle E, we can define a family of laplacians
$$\widehat{D} : \Omega^p(E) \to \Omega^p(E)$$
$$\widehat{D} = \mathrm{d}^\nabla (\mathrm{d}^\nabla)^* + (\mathrm{d}^\nabla)^* \mathrm{d}^\nabla \quad \text{for } \nabla \in P$$

Spectral Flow

Consider a 1-parameter family
$$D_t : \Gamma(E) \to \Gamma(F) \quad 0 \le t \le 1.$$

For each t we have the picture from §4.5, so we have eigenvalues $\lambda_i(t)$ of the operator $D_t^* D_t$, (or of $\widehat{D}_t$).

Theorem 4.25. *The eigenvalues $\lambda_i(t)$ vary smoothly with t.*

[Check: insert plot of $\lambda_i(t)$'s varying with time, some crossing the 0] At any time t, we have the spectrum $\{\lambda_i(t)\}$ distributed on the real line. As t changes, some of the λ_i's may pass through 0. The spectral flow is the number of times this happens (counted with sign).

Definition 4.26. Suppose $\ker D_0 = \ker D_1 = 0$ (so none of the λ_i start or end at 0). The spectral flow of $\{D_t\}$ is defined as
$$\begin{aligned} SF(D_0, D_1) = &\#\{t_k \ : \ \lambda_i(t) \text{ crosses 0 transversly from } - \text{ to } + \text{ at } t = t_k\} \\ &- \#\{t_k \ : \ \lambda_i(t) \text{ crosses 0 transversly from } + \text{ to } - \text{ at } t = t_k\} \end{aligned}$$

[Check: Show that the spectral flow $SF(D_0, D_1)$ is independent of the path.]

Exercise 4.27. For a family parametrized by a manifold P with $\dim P > 1$, the spectral flow is well-defined for generic paths, and is independent of path.

Index

Definition 4.28. With D elliptic as above, the index is
$$\begin{aligned} \operatorname{ind} D &= \dim\ker D - \dim\operatorname{coker} D \\ &= \dim\ker D^* D - \dim\ker DD^* \end{aligned}$$

Recall that we have

$$\ker(D) \oplus \bigoplus_{\lambda_i \neq 0} E_{\lambda_i} \xrightarrow{D} \bigoplus_{\lambda_i \neq 0} F_{\lambda_i} \oplus \ker(D^*).$$

Define the "thickened kernel"

$$\ker_\Lambda D := \ker D \oplus \bigoplus_{|\lambda_i| < \Lambda} E_\lambda.$$

[Check: Proof of something like the index is constant or independent of path. Here is what is in the notes: Define the thickened kernel. ... This is finite dimensional Then $\operatorname{ind} D_t = \dim \ker_\Lambda D_t - \dim \ker_\Lambda D_t^*$ for every Λ not in the spectrum of D since $\ker_\Lambda D \to \ker_\Lambda D^*$ is an isom on $(\ker D)^\perp$ to $(\ker D^*)^\perp$. Then SF picture shows this is independent of t in generic paths $\implies$ is independent of D_p for $p \in P$ for connected P. ($\implies$ index is locally constant, can choose different Λ for different subintervals) – this last part I remember. Choose Λ on a small subinterval of $[0, 1]$ so that no λ crosses it, and the index is the same. Whenever you get to a λ crossing, change Λ to where there is no crossing again on a next subinterval. In this way, you show that something (index) is constant till the end.]

4.7. **Determinant Line Bundle.**

(Reference: Quillen '82)

First consider the linear algebra picture. Let V, W, E, F be finite dimensional vector spaces. We will for the moment use the shorthand

$$\bigwedge(V) := \bigwedge^{\text{top}}(V) = \bigwedge^n(V) \cong \mathbb{C}.$$

Note that

- $\bigwedge(V \oplus W) = \bigwedge V \otimes \bigwedge W$ with basis $(v_1 \wedge \cdots \wedge v_n) \wedge (w_1 \wedge \cdots \wedge w_m) = (v_1 \wedge \cdots \wedge v_n) \otimes (w_1 \wedge \cdots \wedge w_m)$
- $\bigwedge(V^*) = (\bigwedge V)^*$ with basis $v_1^* \wedge \cdots \wedge v_n^*$
- Any linear $\alpha : V \to W$ induces a map $\bigwedge V \to \bigwedge W$ by

  $$v_1 \wedge \cdots \wedge v_n \mapsto \alpha v_1 \wedge \cdots \wedge \alpha v_n$$

  which is an isometry if $\det \alpha \neq 0$. Equivalently, we get an element of the <u>determinant line</u>

  $$\det \alpha \in L := \bigwedge (V^*) \otimes \bigwedge (W).$$

Consider a map $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} : V \oplus E \to W \oplus F$. Then

$$\bigwedge(V \oplus E)^* \otimes \bigwedge(W \oplus F) = \bigwedge V^* \otimes \bigwedge E^* \otimes \bigwedge W \otimes \bigwedge$$
$$\det \alpha = \det \alpha_1 \otimes \det \alpha_2.$$

In particular, if α_2 is an isometry, then this induces an isometry

$$\bigwedge V^* \otimes \bigwedge W \xrightarrow{\det \alpha_1} \bigwedge(V \oplus E)^* \otimes \bigwedge(W \oplus F). \tag{36}$$

Now choose inner products on V, W, and decompose $V = \bigoplus V_{\lambda_i}$, $W = \bigoplus W_{\lambda_i}$ into the λ_i eigenspaces of $\alpha^*\alpha$ and $\alpha\alpha^*$. Write, for $\Lambda \notin \sigma(\alpha) =$ the spectrum of $\alpha^*\alpha$,

$$\ker_\Lambda \alpha := \bigoplus_{|\lambda_i|<\Lambda} V_{\lambda_i}$$
$$\operatorname{coker}_\Lambda \alpha := \ker_\Lambda \alpha^*.$$

Notice α restricts to isomorphims $\alpha_{\lambda_i} : V_{\lambda_i} \longrightarrow W_{\lambda_i}$ for $\lambda_i \neq 0$. Then by (36) we have an isomorphism

$$\bigwedge(\ker\alpha)^* \otimes \bigwedge(\operatorname{coker}\alpha) \cong \bigwedge(\ker_\Lambda \alpha)^* \otimes \bigwedge(\operatorname{coker}_\Lambda \alpha)$$

writing $\ker_\Lambda \alpha = \ker\alpha \oplus E$, where $E = \bigoplus_{0<|\lambda_i|<\Lambda} V_{\lambda_i}$.

Now we consider the vector bundle situation

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Then $\varphi(p) = \dim\ker\alpha_p$ is upper semi-continuous.

Set $\kappa = \inf_M \dim\ker\alpha_p$. We have "jumping loci"

$$J_l := \{p \in M \ : \ \varphi(p) \geq \kappa + l\}.$$

In general $J_1 \neq \varnothing$, so $\ker\alpha$ is not a vector bundle. However, we have:

Definition 4.29. The determinant line bundle of α is

$$\mathscr{L}_\alpha = \bigwedge(\ker\alpha) \otimes \bigwedge(\operatorname{coker}\alpha)^*. \tag{37}$$

Lemma 4.30. *The bundle $\mathscr{L}_\alpha$ is a locally trivial complex line bundle.*

Proof. At a point $p \in M$, choose (local) metrics on V and W and decompose them into eigenspaces of $\alpha^*\alpha$ as above. Fix Λ not in the spectrum $\sigma(\alpha_p^*\alpha_p)$. By the continuous dependence of the eigenvalues, there is a neighborhood U of p, such that if $q \in U$, Λ is also not in $\sigma(\alpha_q^*\alpha_q)$. Then (36) implies that $\mathscr{L}_\alpha|_U$ is trivial. ☐

Example 4.31. This applies in infinite dimensions as well. Let $\{D^\nabla : \nabla \in \mathcal{A}\}$ be a family of elliptic LDO's over a compact manifold M, parameterized by $\mathcal{A}$ a space of connections. We get induced maps $L^2(E) \xrightarrow{D} L^2(F)$, and

$$\mathscr{L} = \det D = \bigwedge(\ker D)^* \otimes \bigwedge(\operatorname{coker} D)$$

forms a complex vector bundle over $\mathcal{A}$

4.8. **Basic analytic constructions of gauge theory.**

Let (M, g) be a compact Riemannian manifold. For a metric vector bundle $E \to M$, the space $W^{k,p}(E) := W^{k,p}(\Gamma(E))$ is a Banach space, and is contained in $C^0(E)$ if $k - \frac{n}{p} > 0$. We also have the multiplication theorem:

$$W^{k_1,p_1}(E) \times W^{k_2,p_2}(E) \to W^{k,p}(E)$$
$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

is continuous if $\left(k_1 - \frac{n}{p_1}\right) + \left(k_2 - \frac{n}{p_2}\right) > \left(k - \frac{n}{p}\right)$.

Lemma 4.32. *For any fiber bundle*

$$\begin{array}{ccc} F & \hookrightarrow & X \\ & & \downarrow \\ & & M \end{array}$$

the space $W^{k,p}(F)$ of $W^{k,p}$ sections is a smooth Banach manifold if $k - \frac{n}{p} > 0$.

[Check: Why, and why a manifold?]

In the setting of gauge theory: Let $P \to M$ be a principal G-bundle, with a compact Lie group G. Fix a smooth reference connection ∇^0, choose k, p with $k \geq 1, k - \frac{n}{p} > -1$, and consider the space of connections

$$\mathcal{A} = \mathcal{A}^{k,p} = \{\nabla^0 + A \;:\; A \in W^{k,p}(T^*M \otimes \operatorname{ad} P)\}.$$

This is an affine Banach space.
Let the gauge group be

$$\mathcal{G} = \mathcal{G}^{k+1,p} = \{\gamma \in W^{k+1,p}(P \times_{\operatorname{Ad}} G)\}.$$

This is a smooth Banach manifold by the lemma, and the group operations

$$\text{multiplication} \quad \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$
$$\text{inversion} \quad \mathcal{G} \to \mathcal{G}, \quad \gamma \mapsto \gamma^{-1}$$

are smooth, which makes $\mathcal{G}$ a Banach Lie group. For $p \in M$, we have the closed Lie subgroup

$$\mathcal{G}_p = \{\gamma \in \mathcal{G} \;:\; \gamma(p) = \operatorname{Id}\}.$$

[Check: all of this]

Main case

On a 4-manifold (or perhaps $n \leq 4$), We might try $\mathcal{A}^{1,2}$ and $\mathcal{G}^{2,2}$ (so that $F = \mathrm{d}A$ and $\int |F|^2$ make sense), but $\mathcal{G}^{2,2}$ is not Banach because we are in the borderline case $\left(k - \frac{n}{p} = 1 - \frac{4}{2} \not> -1\right)$. So instead we use $\mathcal{A}^{2,2}$ and $\mathcal{G}^{3,2}$. (When needed, set $(k,p) = (2,2)$ below.)

I. The curvature map $\nabla \mapsto F^\nabla$ on smooth connections induces a smooth map

$$\mathcal{A}^{k,p} \to W^{k-1,p}\left(\textstyle\bigwedge^2 T^*M \otimes \operatorname{ad} P\right)$$

for $k - \frac{n}{p} > 0$.

Proof. For $\nabla = \nabla^0 + A$, we have the formula $F^\nabla = F^0 + \mathrm{d}^0 A + A \wedge A$. Note

$$\mathrm{d}^0 : W^{k,p}\left(\textstyle\bigwedge^1 T^*M \otimes \operatorname{ad} P\right) \to W^{k-1,p}\left(\textstyle\bigwedge^2 T^*M \otimes \operatorname{ad} P\right)$$

is bounded. Moreover $A \mapsto A \wedge A$ induces

$$W^{k,p} \xrightarrow{\text{diag}} W^{k,p} \times W^{k,p} \longrightarrow W^{k-1,p}$$

which is a composition of a bounded linear map and a bilinear map which is smooth for

$$2\left(k-\frac{n}{p}\right) > k-1-\frac{n}{p} \iff k-\frac{n}{p} > -1$$

□

II. The Yang-Mills energy

$$\mathrm{YM}(\nabla) = \frac{1}{2}\int_M |F|^2 \mathrm{dvol}$$

is a smooth map $\mathcal{A}^{k,p} \to \mathbb{R}$.

Proof. We have $W^{k-1,p} \hookrightarrow L^2$ for $k-1-\frac{n}{p} \geq 0-\frac{n}{2}$ which holds for the above (k,p) when $n \leq 4$. □

III. The group $\mathcal{G}$ acts smoothly on $\mathcal{A}$.

Proof. A gauge transformation $\gamma \in \mathcal{G}$ acts by $A^\gamma = \gamma A \gamma^{-1} - (\nabla^0\gamma)\gamma^{-1}$. This is a combination of the smooth maps

- $\mathcal{G} \to \mathcal{G}$ by $\gamma \mapsto \gamma^{-1}$
- (linear) multiplication $W^{3,2} \times W^{2,2} \longrightarrow W^{2,2}$
- $\mathcal{G} \to W^{2,2}(\mathrm{Ad}\, P)$ by $\gamma \mapsto \nabla^0\gamma$

□

IV. Define the orbit spaces

$$\mathcal{B} = \mathcal{A}/\mathcal{G} \quad \text{and} \quad \mathcal{B}_p = \mathcal{A}/\mathcal{G}_p.$$

(Note that $\mathcal{G}/\mathcal{G}_p \cong G$ so "the difference between these" is finite dimensional.)
For the above (k,p),

Lemma 4.33. *The spaces $\mathcal{B}$ and $\mathcal{B}_p$ are Hausdorff.*

This is not obvious (for quotient spaces). Consider, for example, the action of $\mathbb{R}^*$ on $\mathbb{R}^2$ by $t.(x,y) \mapsto (tx, y/t)$. The equivalence classes are the curves $xy = c$, the origin, and each axis minus the origin. The quotient space is not Hausdorff at the origin. In this case we may have two gauge orbits limiting close to each other.

Proof. The L^2 metric on $\mathcal{A}$

$$\mathrm{dist}\left(\nabla^0 + A, \nabla^0 + B\right) := \left(\int_M |A-B|^2\right)^{\frac{1}{2}} \tag{38}$$

is independent of ∇^0 ($(X+A)-(X+B) = A-B$ for any X), and is $\mathcal{G}$ invariant

$$|A^\gamma - B^\gamma| = \left|\gamma(A-B)\gamma^{-1}\right| = |A-B|, \tag{39}$$

so it descends to a function

$$d([A],[B]) := \inf_{\gamma\in\mathcal{G}} \|A - \gamma B\|_{L^2} = \inf_{\gamma\in\mathcal{G}} \left\|\gamma^{-1}A - B\right\|_{L^2} \tag{40}$$

on the quotient space of gauge orbits $[A]$ of $\nabla^0 + A$. The claim is that if $d([A],[B]) = 0$, this implies that $[A] = [B]$, so that d is a metric. □

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